Finite Element Approximations to Surfaces of Prescribed Variable Mean Curvature

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Abstract: We give an algorithm for finding finite element approximations to surfaces of prescribed variable mean curvature, which span a given boundary curve. We work in the parametric setting and prove optimal estimates in the H^1 norm. The estimates are verified computationally.

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1 Introduction

We are interested in finding disc-like surfaces in $\mathbb{R}^3$ with a given smooth, embedded and possibly knotted boundary curve Γ, and mean curvature at each point u prescribed by a smooth function $H(u)$. The classical Plateau problem is the case $H \equiv 0$. Physical processes which can be modelled by surfaces of prescribed mean curvature include hanging drops, soap films and the limiting behaviour of phase transition interfaces in the Van der Waals-Cahn-Hilliard theory.

We expect that the methods developed here can be extended to treat a variety of geometric problems.

In order to explain our approach we introduce the following notation. Let $D \subseteq \mathbb{R}^2$ be the unit disc and let

$$\mathcal{S}(\Gamma) = \{u \in H^1 \cap C^0(\overline{D}; \mathbb{R}^3) \mid u|_{\partial D} \text{ monotonely parametrises } \Gamma\}. \tag{1}$$

Suppose $Q : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^2 vector field and let $\operatorname{div} Q = 3H$. For $u \in \mathcal{S}(\Gamma)$ define the energy functional by

$$\mathcal{E}(u) = \frac{1}{2}\int_D |\nabla u|^2 + \frac{2}{3}\int_D Q(u) \cdot u_x \wedge u_y. \tag{2}$$

If $Q = \alpha u$ for some constant α then $H = \alpha$ and the second ("volume") term is 2α times the volume of the cone from the origin to the surface u.

If u is stationary for $\mathcal{E}$ then u conformally parametrises a surface spanning Γ with mean curvature $H(u)$. (For justification of this and the other claims here and in the following paragraph see [St1] and [DHKW].) If Γ has a $C^{k,\alpha}$ monotone parametrisation for some $k \geq 2$ then u is also $C^{k,\alpha}$. The following Euler-Lagrange system of equations is satisfied by u:

$$\begin{aligned} \Delta u &= 2H(u)\, u_x \wedge u_y, \\ |u_x| &= |u_y|, \quad u_x \cdot u_y = 0. \end{aligned} \tag{3}$$

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Moreover, if $u \in \mathcal{S}(\Gamma)$ satisfies the above system of equations then u is stationary for $\mathcal{E}$. Notice the nonlinearity in the equations and also in the boundary condition on members of $\mathcal{S}(\Gamma)$. Conversely, if $v \in \mathcal{S}(\Gamma)$ is a surface spanning Γ with mean curvature $H(v)$ then any conformal reparametrisation u of the surface given by v will belong to $\mathcal{S}(\Gamma)$ and is stationary for $\mathcal{E}$. Thus an advantage of working with the functional $\mathcal{E}$ is that one obtains *conformal* parametrisations of the prescribed mean curvature surfaces.

In this paper we give a method for finding finite element approximations u_h to functions u stationary for $\mathcal{E}$, and prove the estimate

$$\|u - u_h\|_{H^1(D)} \leq ch, \tag{4}$$

where c depends on u.

In [DH3, DH4] we treated the case $H \equiv 0$. The methods there do not extend to the present problem, as we discuss later, and so we develop a new and more general framework. We note that for non-zero H there may not even exist a minimiser of $\mathcal{E}$, but if there is one then there typically also exists a corresponding unstable stationary point (corresponding in some sense to the cubic nature of the second integrand in $\mathcal{E}$).

The difficulty of the problem arises through the nonlinearities of the class of competing functions and of the Euler Lagrange equations, as remarked before. However, we can linearise the class of competing functions, at the price of introducing further nonlinear (more precisely, non quadratic) features into the energy functional.

For this purpose we let S^1 denote the unit circle in $\mathbb{R}^2$, and consider a fixed monotone $C^{k,\alpha}$ parametrisation

$$\gamma : S^1 \to \Gamma \tag{5}$$

for suitable $k \geq 2$. It will be convenient to distinguish between the circle S^1 and the boundary ∂D of the unit disc. Boundary maps $g : \partial D \to \Gamma$ will be written in the form $g = \gamma \circ s$ where $s : \partial D \to S^1$.

In the case $Q = 0$ of minimal surfaces, stationary points for $\mathcal{E}$ are harmonic by (3). (We emphasise however that we are *not* working with Dirichlet boundary conditions; the (free type) boundary condition is $u|_{\partial D} : \partial D \to \Gamma$ with $u|_{\partial D}$ monotone.) In this case it is permissible to look for stationary solutions for $\mathcal{E}$ within the subclass of harmonic functions from $\mathcal{S}(\Gamma)$. The nonlinearity in this subclass due to the boundary condition was removed in [DH3, DH4] by working with a suitable (affine) space of maps $s : \partial D \to S^1$ (having winding number one) and by considering the unique *harmonic extension* $\Phi(\gamma \circ s)$ of $\gamma \circ s$ as well as its discrete analogue, c.f. (3.1).

The analogous approach here would be to consider *H-harmonic functions* (i.e. solutions of the first equation in (3)) with boundary data $\gamma \circ s$. But such extensions may not exist, nor be unique if they do exist and the extension operator is itself nonlinear and its first and second variations are quite complicated. In [DH2] we did treat the case of finite element approximations to solutions of the first equation in (3) with *Dirichlet* boundary data. The analytic theory of such functions is treated in [St1].

In this paper we develop instead an idea from [St1, Section IV.4] and linearise the class of competing surfaces by considering a suitable affine space of pairs (s, φ) with s as before and with $\varphi : D \to \mathbb{R}^3$ satisfying $\varphi|_{\partial D} = 0$, and by taking the corresponding function

$$u = \Phi(\gamma \circ s) + \varphi =: u_0 + \varphi. \tag{6}$$

Thus u is decomposed into the sum of an harmonic function with the same boundary data and a function with zero boundary data. Note that if u is sufficiently smooth and $u|_{\partial D}$ is one-one, then s and φ are uniquely determined by u from

$$\gamma \circ s = u|_{\partial D}, \quad \varphi = u - \Phi(\gamma \circ s), \tag{7}$$

while u is clearly uniquely determined by (s, φ). In [DH3, DH4] we in effect made the further restriction $\varphi \equiv 0$, but of course that is not possible here. Here we need to consider the class of *all* sufficiently smooth functions $u : D \to \mathbb{R}^3$ for which $u|_{\partial D}$ is a (monotone) parametrisation of Γ. The precise class of functions (s, φ) is discussed in the following Section.

The energy functional is next defined by

$$E(s, \varphi) = \frac{1}{2} \int_D |\nabla u|^2 + \frac{2}{3} \int_D Q(u) \cdot u_x \wedge u_y, \tag{8}$$

with u as in (6). The first and second variations are computed in (21) and (22). One seeks solutions of $E'(s, \varphi) = 0$ and then the required prescribed mean curvature surface is given by u as in (6).

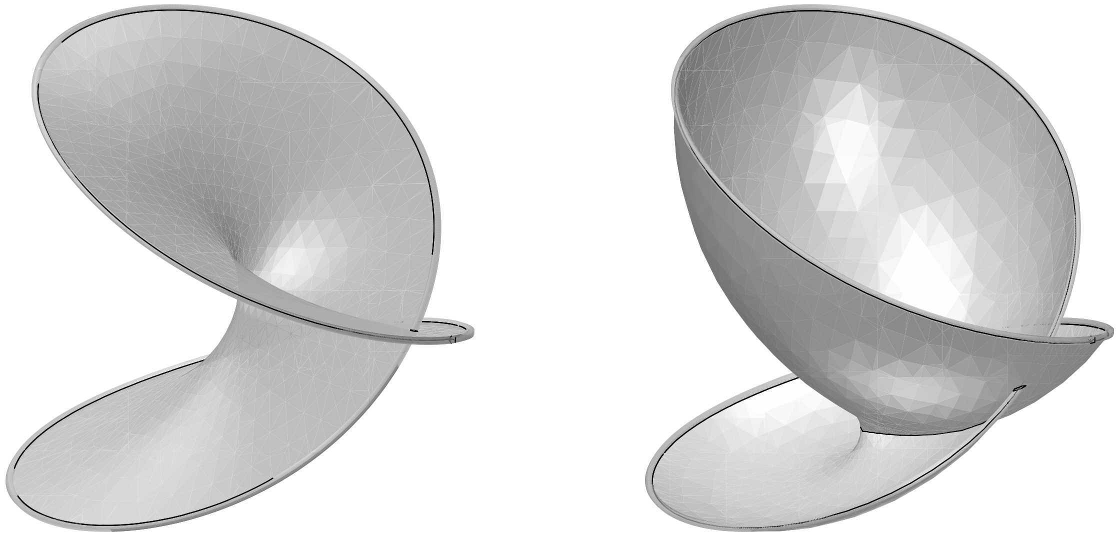

Figure 1: Solutions of the Plateau Problem for the same boundary curve with $H = 0$ (left) and $H = 0.8$ (right).

For numerical approximations we consider regular triangulations D_h of D with grid size h and continuous piecewise linear finite elements (s_h, φ_h). The φ_h are piecewise linear on D_h with zero boundary values. The $s_h : \partial D \to S^1$ are piecewise linear (on ∂D not ∂D_h) in the sense of arc length; this choice of domain for s_h has the technical advantage that the s_h form a (affine) subspace of the s. However, the piecewise linear "interpolant" $I_h(\gamma \circ s_h)$ is defined on ∂D_h (not on ∂D) in the natural manner, agrees with $\gamma \circ s_h$ on the boundary nodes, and in particular maps boundary nodes onto Γ.

In analogy with (6) we define the piecewise linear function

$$u_h = \Phi_h(I_h(\gamma \circ s_h)) + \varphi_h, \tag{9}$$

where Φ_h is the discrete harmonic extension operator defined for piecewise linear g_h on ∂D_h by

$$\Phi_h(g_h) = v_h \quad \Leftrightarrow \quad \begin{cases} v_h = g_h & \text{on } \partial D_h, \\ \int_{\Omega_h} \nabla v_h \nabla \psi_h = 0, \end{cases} \tag{10}$$

for all ψ_h continuous and piecewise linear on D_h and zero on ∂D_h.

The discrete energy functional is then defined by ("$\wedge$" is cross product)

$$E_h(s_h, \varphi_h) = \frac{1}{2} \int_{D_h} |\nabla u_h|^2 + \frac{2}{3} \int_{D_h} Q(u_h) \cdot u_{hx} \wedge u_{hy}. \tag{11}$$

Setting E_h' to be the first variation of E_h, solutions of $E_h'(s_h, \varphi_h) = 0$ can be found by standard Newton procedures. The existence of such a (unique) discrete solution in a sufficiently small neighbourhood of a smooth solution (s, φ) of the original problem is proved by an inverse function theorem argument which also leads to estimates of the form

$$|s - s_h|_{H^{1/2}(\partial D)} \leq ch, \quad |\varphi - \varphi_h|_{H^1(D)} \leq ch,$$

and from which (4) then follows.

We remark that our method has the additional advantage of producing particularly good triangulations of the resulting surfaces, namely a discrete conformal mapping from the initial domain.

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2 The Variational Problem

In this and the following section we make precise the ideas from the Introduction.

Throughout the paper Γ *is an embedded curve in* $\mathbb{R}^3$ *with monotone* C^4 *parametrisation* $\gamma : S^1 \to \Gamma$. *We assume* Q *is a* C^2 *vector field on* $\mathbb{R}^3$ *and the prescribed mean curvature function* H *is given by* $\operatorname{div} Q = 3H$. Less regularity is required for many of the estimates, as we will indicate.

2.1 Function Spaces

The set $\mathcal{H}$ of functions $s : \partial D \to S^1$ natural for the problem as treated here (see the Introduction) is given in (14). But it is convenient to first describe the space H of allowable variations ξ, see (12) and (13). Whereas $s : \partial D \to S^1$, it is natural to take $\xi : \partial D \to \mathbb{R}$. In particular such variations will form a vector space, unlike the set of s which form only an affine space. With the natural notion of addition one has $s + \xi \in \mathcal{H}$; see also (14) and the subsequent comments.

It is well known that there is a three parameter family of conformal parametrisations u from the unit disc to any sufficiently smooth disc-like surface. A unique conformal representative is usually selected by specifying $u(p_i)$ for fixed points $p_1, p_2, p_3 \in \partial D$. Here it is required instead that with $u|_{\partial D} = \gamma \circ s$ as in (7) then s should satisfy three integral conditions, see (14) and also (12), (13). The justification for the existence and uniqueness of a conformal reparametrisation of any given u satisfying these conditions follows from a topological fixed point argument, see the appendix to the preprint version of [DH1].

Thus we first define

$$H = \left\{ \xi : \partial D \to I\!R \mid \xi \in H^{1/2} \text{ and (13) is satisfied} \right\}, \tag{12}$$

$$\int_0^{2\pi} \xi(\theta)\, d\theta = 0, \ \int_0^{2\pi} \xi(\theta) \cos\theta\, d\theta = 0, \ \int_0^{2\pi} \xi(\theta) \sin\theta\, d\theta = 0. \tag{13}$$

Note that the semi-norm $|\cdot|_{H^{1/2}}$ and the norm $\|\cdot\|_{H^{1/2}}$ are equivalent on H by the first condition in (13).

Here and elsewhere we identify points $e^{i\theta} \in \partial D$ (or S^1) with $\theta \in [0, 2\pi]$, functions $\xi : \partial D \to \mathbb{R}$ with 2π-periodic functions, and frequently write θ for $e^{i\theta}$.

The affine space of appropriate s is defined by

$$\mathcal{H} = \{ s : \partial D \to S^1 \mid s = \mathrm{id} + \sigma, \ \sigma \in H \} \tag{14}$$

where id : $\partial D \to S^1$ is the "identity" map $\mathrm{id}(\theta) = \theta$ or equivalently, $\mathrm{id}(e^{i\theta}) = e^{i\theta}$. The sum in the above definition is defined in the natural way by $s(e^{i\theta}) = e^{i(\theta+\sigma(e^{i\theta}))}$, and should be thought of as mapping the point on ∂D with angle θ to the point on S^1 with angle $\theta + \sigma(\theta)$.

2.2 Energy Functional and Variations

The energy functional E is defined by

$$E(s,\varphi) = \frac{1}{2}\int_D |\nabla u|^2 + \frac{2}{3}\int_D Q(u)\cdot u_x \wedge u_y, \tag{15}$$

where

$$(s,\varphi) \in (\mathcal{H} \cap C^0)(\partial D; S^1) \times (H_0^1 \cap C^0)(\overline{D}; I\!R^3), \tag{16}$$

$$u = \Phi(\gamma \circ s) + \varphi =: u_0 + \varphi, \quad \Phi : C^0(\partial D; I\!R^3) \to C^0(\overline{D}; I\!R^3) \tag{17}$$

and Φ is the harmonic extension operator.

Variations of (s,φ) are given by

$$(\xi,\psi) \in (H \cap C^0)(\partial D; I\!R) \times (H_0^1 \cap C^0)(\overline{D}; I\!R^3), \tag{18}$$

and the corresponding first and second variations of u are denoted by

$$v = \frac{d}{dt}\Big|_{t=0} u(s+t\xi, \varphi+t\psi) = \Phi(\xi\, \gamma' \circ s) + \psi =: v_0 + \psi, \tag{19}$$

$$w = \frac{d^2}{dt^2}\Big|_{t=0} u(s+t\xi, \varphi+t\psi) = \Phi(\xi^2\, \gamma'' \circ s). \tag{20}$$

The first and second variations of E can then formally be computed to give

$$E'(s,\varphi)(\xi,\psi) = \int_D \nabla u \cdot \nabla v + 2\int_D H(u) v \cdot u_x \wedge u_y \tag{21}$$

$$\begin{aligned} E''(s,\varphi)(\xi,\psi)^2 &= \int_D |\nabla v|^2 + \int_D \nabla u \cdot \nabla w + 2\int_D H'(u)\cdot v\ v \cdot u_x \wedge u_y \\ &+ 2\int_D H(u)\big(w\cdot u_x \wedge u_y + v\cdot(v_x \wedge u_y + u_x \wedge v_y)\big). \end{aligned} \tag{22}$$

The first variation is obtained using an integration by parts. Some decoupling of terms occurs since

$$\int_D \nabla u \cdot \nabla v = \int_D \nabla u_0 \cdot \nabla v_0 + \int_D \nabla\varphi \cdot \nabla\psi, \tag{23}$$

$$\int_D |\nabla v|^2 + \nabla u \cdot \nabla w = \int_D |\nabla v_0|^2 + \int_D |\nabla \psi|^2 + \int_D \nabla u_0 \cdot \nabla w. \tag{24}$$

There are significant technical difficulties concerning the appropriate spaces in which to work in order to establish existence and regularity results for stationary solutions of the above variational problem. For example, if $H(u)$ is constant then E is continuous and bounded on $\mathcal{H} \times H_0^1$ but C^1 only on the smaller space $(\mathcal{H} \cap C^0) \times (H_0^1 \cap C^0)$, see [St1] and [DH3, DH4] (for $H = 0$). In the case of variable $H(u)$ the situation is even more complicated, see [St2, Section 4].

It is not difficult to show for $c = c(\|s\|_{C^2}, \|\varphi\|_{C^2})$ that

$$\begin{aligned} |E'(s,\varphi)(\xi,\psi)| &\le c(|\xi|_{H^{1/2}} + |\psi|_{H^1}), \\ |E''(s,\varphi)(\xi,\psi)^2| &\le c(|\xi|^2_{H^{1/2}} + |\psi|^2_{H^1}), \end{aligned} \tag{25}$$

and hence that $E'(s,\varphi)$ and $E''(s,\varphi)$ can be considered as bounded linear and bilinear functionals respectively on $H\times H_0^1$ (c.f. [DH3, Proposition 3.9]; the additional terms in the current setting cause no extra difficulties). Note that $H\times H_0^1$ is an inner product space with norm

$$|(\xi,\psi)|_{H\times H^1} = (|\xi|^2_{H^{1/2}} + |\psi|^2_{H^1})^{1/2}. \tag{26}$$

Definition 2.1. We say $u=\Phi(\gamma\circ s)+\varphi$ is a (conformally parametrised) surface spanning Γ with prescribed mean curvature H if $(s,\varphi)\in\mathcal{H}\times H_0^1$ is stationary for E, i.e. if

$$E'(s,\varphi)(\xi,\psi)=0 \tag{27}$$

for all $(\xi,\psi)\in H\times H_0^1$. We say u is non-degenerate if

$$\lambda = \inf_{(\xi,\psi)}\sup_{(\eta,\chi)} E''(s,\varphi)\big((\xi,\psi),(\eta,\chi)\big) > 0, \tag{28}$$

where $(\xi,\psi)\in H\times H_0^1$, $|\xi|^2_{H^{1/2}}+|\psi|^2_{H^1}=1$, and similarly for (η,χ).

We will assume u,s,φ are all C^2.

The existence of such u (which are monotone on ∂D) is established for constant H in Struwe [St1] where the equivalence with the classical formulation (3) is also shown. Struwe shows by Morse theory methods the existence of both "small" and "large" solutions, and in particular of unstable solutions. See also [DH3, DH4] for further discussion and examples in the minimal surface case. The existence of solutions for variable H in the present setting is discussed in [St2, Section 4]. The regularity of solutions is established in [St1], with a gap filled by [Imb], by reducing to the classical formulation and using earlier regularity results of Hildebrandt [Hil], Nitsche [N1], Jäger [Ja] and Heinz [He]. A result sufficient for our purposes is that if γ is $C^{k,\alpha}$ and Q is $C^{k+1,\alpha}$ for $k\ge 2$, then s, φ are $C^{k,\alpha}$ and $u\in C^{k,\alpha}(\overline{D})$.

2.3 Positive and Negative Eigenspaces for the Second Variation

Since $E''(s,\varphi)$ is symmetric, bounded and bilinear on $H\times H_0^1$ it induces a bounded self-adjoint linear operator

$$\nabla^2 E(s,\varphi): H\times H_0^1\to H\times H_0^1 \tag{29}$$

whose eigenvalues are real, and this gives a splitting of $H\times H_0^1$ into positive and negative eigenspaces (the null space being trivial as $\lambda>0$), denoted

$$\begin{aligned} H\times H_0^1 &= (H\times H_0^1)^-\oplus(H\times H_0^1)^+,\\ (\xi,\psi) &= (\xi,\psi)^- + (\xi,\psi)^+. \end{aligned} \tag{30}$$

The non-degeneracy condition (28) is equivalent to

$$\begin{aligned} E''(s,\varphi)&((\xi,\psi),(\xi,\psi)^+-(\xi,\psi)^-)\\ &= E''(s,\varphi)((\xi,\psi)^+)^2 - E''(s,\varphi)((\xi,\psi)^-)^2\\ &\ge \lambda(|\xi|^2_{H^{1/2}}+|\psi|^2_{H^1}) \end{aligned} \tag{31}$$

for all $(\xi,\psi)\in H\times H_0^1$. It is also equivalent to the invertibility of $E''(s,\varphi)$ regarded as a map $E''(s,\varphi): H\times H_0^1\to(H\times H_0^1)'$ (the dual space) , with

$$\big\|\big(E''(s,\varphi)\big)^{-1}\big\|\le\lambda^{-1}. \tag{32}$$

Finally *we assume* $(H\times H_0^1)^-$ *is finite dimensional and that if* $(\xi,\psi)\in(H\times H_0^1)^-$ *then for some* ν,

$$\|\xi\|_{H^{3/2}}+\|\psi\|_{H^2}\le\nu(|\xi|_{H^{1/2}}+|\psi|_{H^1}). \tag{33}$$

If the negative eigenspace is trivial, i.e. the stationary point (s,φ) is *stable*, then non-degeneracy is usually called *strict stability.*

The finite dimensionality of $(H\times H^1_0)^-$ follows by standard arguments from a Garding type inequality for $E''(s,\varphi)$, c.f. [St1, Proposition II.5.6]. Regularity follows as in [St1, Proposition II.5.6] and [Imb], since each member of $(H\times H^1_0)^-$ is a finite sum of solutions of certain elliptic partial differential equations. It is consistent with these results to take $\nu=\nu(\|\gamma\|_{C^4},\|s\|_{C^2},M,\|\varphi\|_{C^2},\|(\gamma')^{-1}\|_{C^0},d)$ where d is the dimension of $(H\times H^1_0)^-$.

3 The Discrete Variational Problem

3.1 Discrete Function Spaces

Let $\mathcal{G}_h$ be a quasi-uniform triangulation of D with grid size comparable to h. Let

$$\begin{aligned} D_h=\bigcup\{\,G\mid G\in\mathcal{G}_h\,\},\quad \partial D_h=\bigcup\{\,E_j\mid 1\le j\le M\,\},\\ \mathcal{B}_h=\{\theta_1,\dots,\theta_M\}\ \text{(the set of boundary nodes)},\quad \theta_{M+1}=\theta_1.\end{aligned}\tag{34}$$

The projection $\pi\colon\partial D\to\partial D_h$ is defined for $0\le t\le 1$, $1\le j\le M$ by

$$\pi\big(e^{i((1-t)\theta_j+t\theta_{j+1})}\big)=(1-t)e^{i\theta_j}+te^{i\theta_{j+1}}.\tag{35}$$

We define the discrete spaces:

$$\begin{aligned} X_h&=\big\{\,u_h:D_h\to I\!R^3\mid u_h\ \text{continuous and piecewise linear}\big\}\\ X_{h0}&=\{\,\psi_h\in X_h\mid \psi_h=0\ \text{on}\ \partial D_h\,\}\\ H_h&=\{\,\xi_h:\partial D\to I\!R\mid \xi_h\ \text{continuous p.l., and satisfies (13)}\,\}\\ \mathcal{H}_h&=\big\{\,s_h:\partial D\to S^1\mid\ s_h=\mathrm{id}+\sigma_h\ \text{for some}\ \sigma_h\in H_h\,\big\}.\end{aligned}\tag{36}$$

The domain of the discrete energy functional will be the affine space $\mathcal{H}_h\times X_{h0}$, which by extending members of X_{h0} to be zero on $D\setminus D_h$ is a subspace of the domain of E. The domain of the set of variations for $(s_h,\varphi_h)\in\mathcal{H}_h\times X_{h0}$ is the vector space $H_h\times X_{h0}$, which is a subspace of the corresponding space of variations (18) in the smooth setting.

For $f\in C^0(\partial D;I\!R^3)$ we define the two *interpolation functions* I_hf and $I_h^{\partial D}f$, whose domains are ∂D_h and ∂D respectively, by

$$\begin{aligned} I_hf\big((1-t)e^{i\theta_j}+te^{i\theta_{j+1}}\big)&=I_h^{\partial D}f\big(e^{i((1-t)\theta_j+t\theta_{j+1})}\big)\\ &=(1-t)f(e^{i\theta_j})+tf(e^{i\theta_{j+1}}),\end{aligned}\tag{37}$$

where $0\le t\le 1$, $1\le j\le M$. Clearly,

$$I_h^{\partial D}f=I_hf\circ\pi.\tag{38}$$

For $u\in C^0(\overline{D};\mathbb{R}^3)$ define the *piecewise linear interpolation function* $I_hu\in X_h$ in the usual way.

Concerning maps $s:\partial D\to S^1$ and $\xi:\partial D\to\mathbb{R}$, we linearly interpolate and then L^2 project onto H_h. The resulting linear operators p_h preserve the integral conditions (13) and still satisfy the usual estimates, see [DH4, Proposition 5.2]. In particular, for $\xi\in H\cap C^0$, $k=1,\frac{3}{2},2$, and $s\in\mathcal{H}\cap C^0$,

$$\|\xi-p_h\xi\|_{H^{1/2}}\le ch^{k-1/2}\|\xi\|_{H^k},\tag{39}$$

$$\begin{aligned}\|s-p_hs\|_{C^{0,1}}\le ch\|s\|_{C^2},\quad \|p_hs\|_{C^{0,1}}\le c\|s\|_{C^{0,1}},\\ \|s-p_hs\|_{C^0}\le ch^2\|s\|_{C^2},\quad \|s-p_hs\|_{C^0}\le ch\|s\|_{C^1}.\end{aligned}\tag{40}$$

3.2 The Discrete Energy Functional and Its Variations

If $f_h : \partial D_h \to I\!R^3$ is continuous and piecewise linear, its *discrete harmonic extension* $\Phi_h f_h \in X_h$ is defined by

$$\begin{aligned} \int_{D_h} \nabla(\Phi_h f_h) \cdot \nabla \psi_h = 0 \quad \text{for all } \psi_h \in X_{h0}, \\ \Phi_h f_h = f_h \quad \text{on} \quad \partial D_h. \end{aligned} \tag{41}$$

For $(s_h, \varphi_h) \in \mathcal{H}_h \times X_{h0}$, the *discrete energy functional* is defined by

$$E_h(s_h, \varphi_h) = \frac{1}{2} \int_{D_h} |\nabla u_h|^2 + \frac{2}{3} \int_{D_h} Q(u_h) \cdot u_{hx} \wedge u_{hy}, \tag{42}$$

$$u_h = \Phi_h I_h(\gamma \circ s_h) + \varphi_h =: u_{h0} + \varphi_h. \tag{43}$$

Note that E_h is *not* the restriction of E to $\mathcal{H}_h \times X_{h0}$.

The first and second variations of u_h in the direction $(\xi_h, \psi_h) \in H_h \times X_{h0}$ are easily computed to be

$$\begin{aligned} v_h &= \frac{d}{dt}\Big|_{t=0} u_h(s_h + t\xi_h, \varphi_h + t\psi_h) = \Phi_h I_h(\xi_h \gamma' \circ s_h) + \psi_h \\ &=: v_{h0} + \psi_h, \end{aligned} \tag{44}$$

$$w_h = \frac{d^2}{dt^2}\Big|_{t=0} u_h(s_h + t\xi_h, \varphi_h + t\psi_h) = \Phi I_h(\xi_h^2 \gamma'' \circ s_h). \tag{45}$$

Denoting differentiation along ∂D_h by the subscript τ, the first and second variations of E_h are readily computed to be

$$\begin{aligned} E_h'(s_h, \varphi_h)(\xi_h, \psi_h) = &\int_{D_h} \nabla u_h \cdot \nabla v_h + 2 \int_{D_h} H(u) v \cdot u_x \wedge u_y \\ &- \frac{2}{3} \int_{\partial D_h} Q(u_h) \cdot v_h \wedge u_{h\tau}, \end{aligned} \tag{46}$$

$$\begin{aligned} E_h''(s_h, \varphi_h)(\xi_h, \psi_h)^2 = &\int_{D_h} |\nabla v_h|^2 + \int_{D_h} \nabla u_h \cdot \nabla w_h \\ &+ 2 \int_{D_h} H'(u_h) \cdot v_h \; v_h \cdot u_{hx} \wedge u_{hy} \\ &+ 2 \int_{D_h} H(u_h) \big(w_h \cdot u_{hx} \wedge u_{hy} + v_h \cdot (v_{hx} \wedge u_{hy} + u_{hx} \wedge v_{hy}) \big) \\ &- \frac{2}{3} \int_{\partial D_h} (Q'(u_h) v_h) \cdot v_h \wedge u_{h\tau} \\ &- \frac{2}{3} \int_{\partial D_h} Q(u_h) \cdot (w_h \wedge u_{h\tau} + v_h \wedge v_{h\tau}). \end{aligned} \tag{47}$$

The first variation is obtained from an integration by parts, but unlike the smooth case one also obtains a boundary term. Note the splitting of terms

$$\int_{D_h} \nabla u_h \cdot \nabla v_h = \int_{D_h} \nabla u_{h0} \cdot \nabla v_{h0} + \int_{D_h} \nabla \varphi_h \cdot \nabla \psi_h \tag{48}$$

$$\begin{aligned} \int_{D_h} |\nabla v_h|^2 + \nabla u_h \cdot \nabla w_h = &\int_{D_h} |\nabla v_{h0}|^2 + \int_{D_h} |\nabla \psi_h|^2 \\ &+ \int_{D_h} \nabla u_{h0} \cdot \nabla w_h. \end{aligned} \tag{49}$$

Definition 3.1. The function $u_h = \Phi_h I_h(\gamma \circ s_h) + \varphi_h$ is a discrete (conformally parametrised) surface spanning Γ with discrete prescribed mean curvature given by H if $(s_h, \varphi_h) \in \mathcal{H}_h \times X_{h0}$ is stationary for E_h, i.e. if

$$E'_h(s_h, \varphi_h)(\xi_h, \psi_h) = 0 \quad \text{for all } (\xi_h, \psi_h) \in H_h \times X_{h0}. \tag{50}$$

We say u_h is non-degenerate with non-degeneracy constant λ^* if

$$\lambda^* = \inf_{(\xi_h, \psi_h)} \sup_{(\eta_h, \chi_h)} E''_h(s_h, \varphi_h)\big((\xi_h, \psi_h), (\eta_h, \chi_h)\big) > 0, \tag{51}$$

where $(\xi_h, \psi_h) \in H_h \times X_{h0}$, $|\xi_h|^2_{H^{1/2}} + |\psi_h|^2_{H^1} = 1$, similarly for (η_h, χ_h).

Note that we do not require monotonicity of the discrete boundary map.

3.3 Discrete Approximations to the Eigenspace Decomposition

For $(\xi_h, \psi_h) \in H_h \times X_{h0}$ and (s, φ) stationary and non-degenerate for E, we define the projection of the decomposition (30):

$$\begin{aligned} (\xi_h, \psi_h)^{(-)} &= p_h(\xi_h, \psi_h)^- \in H_h \times X_{h0}, \\ (\xi_h, \psi_h)^{(+)} &= p_h(\xi_h, \psi_h)^+ \in H_h \times X_{h0}. \end{aligned} \tag{52}$$

Thus

$$(\xi_h, \psi_h) = (\xi_h, \psi_h)^{(-)} + (\xi_h, \psi_h)^{(+)}. \tag{53}$$

Note that $(\xi_h, \psi_h)^-$ and $(\xi_h, \psi_h)^+$ do not normally belong to $H_h \times X_{h0}$, in particular the first is a pair of smooth functions. However, if $(H \times H^1_0)^-$ is trivial then $(\xi_h, \psi_h) = (\xi_h, \psi_h)^+ = (\xi_h, \psi_h)^{(+)}$.

The decomposition (53) is "almost orthogonal" since

$$\begin{aligned} |(\xi_h, \psi_h)^- - (\xi_h, \psi_h)^{(-)}|_{H^{1/2} \times H^1} &\le ch|(\xi_h, \psi_h)|_{H^{1/2} \times H^1}, \\ |(\xi_h, \psi_h)^+ - (\xi_h, \psi_h)^{(+)}|_{H^{1/2} \times H^1} &\le ch|(\xi_h, \psi_h)|_{H^{1/2} \times H^1}, \end{aligned} \tag{54}$$

where c has the same dependencies as in (33). To see this, note from (39) and (33) that

$$\begin{aligned} |(\xi_h, \psi_h)^- - (\xi_h, \psi_h)^{(-)}|_{H^{1/2} \times H^1} &= |(\xi_h, \psi_h)^- - p_h(\xi_h, \psi_h)^-|_{H^{1/2} \times H^1} \\ &\le ch|(\xi_h, \psi_h)^-|_{H^{3/2} \times H^2} \le ch|(\xi_h, \psi_h)^-|_{H^{1/2} \times H^1}. \end{aligned}$$

This gives the first inequality and the second then follows since

$$(\xi_h, \psi_h)^+ - (\xi_h, \psi_h)^{(+)} = (\xi_h, \psi_h)^{(-)} - (\xi_h, \psi_h)^-.$$

It follows from (54) and the orthogonal decomposition (30) that

$$\begin{aligned} |(\xi_h, \psi_h)^{(-)}|_{H^{1/2} \times H^1} &\le (1 + ch)|(\xi_h, \psi_h)^-|_{H^{1/2} \times H^1} \\ &\le (1 + ch)|(\xi_h, \psi_h)|_{H^{1/2} \times H^1} \\ |(\xi_h, \psi_h)^{(+)}|_{H^{1/2} \times H^1} &\le (1 + ch)|(\xi_h, \psi_h)^+|_{H^{1/2} \times H^1} \\ &\le (1 + ch)|(\xi_h, \psi_h)|_{H^{1/2} \times H^1}. \end{aligned} \tag{55}$$

4 Estimates for Smooth and Discrete Functions and their Variations

In this section we estimate the difference between the function u corresponding to an arbitrary smooth pair (s, φ), and the discrete function u_h corresponding to the discrete interpolant pair (s_h, φ_h). Estimates comparing the first and second variations of u and u_h are also obtained. Finally we obtain estimates comparing variations at a discrete pair $(s_h + \eta_h, \varphi_h + \psi_h)$ near the interpolant pair (s_h, φ_h).

4.1 Notation

For future reference we gather together notation for this and the next section.

The domain for norms and semi-norms over D, D_h and ∂D_h will usually be indicated explicitly, while those over ∂D will not. The main exception is that $|\psi_h|_{H^1}$ will be written for $|\psi_h|_{H^1(D_h)}$.

For $s \in \mathcal{H}$ it will be convenient to use the notation

$$\|s\| = 1 + \|\sigma\| \tag{56}$$

where $s = \mathrm{id} + \sigma$, for various norms on σ. Note that $\|s\| \geq 1$.

Following (17), (43), (19), (20), (44) and (45), we define the following interpolants and first and second variations. Motivation for considering these particular quantities and for their estimates is given prior to the relevant lemmas.

$$\begin{aligned} s_h &= p_h s, \quad \varphi_h = I_h \varphi, \\ u &= \Phi(\gamma \circ s) + \varphi =: u_0 + \varphi, \\ u_h &= \Phi_h I_h(\gamma \circ s_h) + \varphi_h =: u_{h0} + \varphi_h. \end{aligned} \tag{57}$$

$$\begin{aligned} v &= \Phi(\xi_h \gamma' \circ s) + \psi_h =: v_0 + \psi_h, \\ v_h &= \Phi_h I_h(\xi_h \gamma' \circ s_h) + \psi_h =: v_{h0} + \psi_h, \\ w &= \Phi(\xi_h^2 \gamma'' \circ s), \\ w_h &= \Phi_h I_h(\xi_h^2 \gamma'' \circ s_h), \end{aligned} \tag{58}$$

$$\begin{aligned} \overline{u}_h &= \Phi_h I_h(\gamma \circ (s_h + \eta_h)) + (\varphi_h + \chi_h) = \overline{u}_{h0} + (\varphi_h + \chi_h), \\ \overline{v}_h &= \Phi_h I_h(\xi_h \gamma' \circ (s_h + \eta_h)) + \psi_h = \overline{v}_{h0} + \psi_h, \\ \overline{w}_h &= \Phi_h I_h(\xi_h^2 \gamma'' \circ (s_h + \eta_h)). \end{aligned} \tag{59}$$

4.2 Preliminary Estimates

The following results for comparing harmonic and discrete harmonic extensions will be used often.

Proposition 4.1. *If $f \in H^r(\partial D, I\!R^n)$ where $r = 1, 3/2$ then*

$$|\Phi(f) - \Phi_h I_h(f)|_{H^1(D_h)} \leq c h^{r-1/2} |f|_{H^r}, \tag{60}$$

$$|\Phi_h I_h(f)|_{H^1(D_h)} \leq |f|_{H^{1/2}} + c h^{r-1/2} |f|_{H^r}, \tag{61}$$

$$\|\Phi(f) - \Phi_h I_h(f)\|_{L^2(D_h)} \leq c h^{r+1/2} |f|_{H^r} + c \|f - I_h^{\partial D} f\|_{L^2}, \tag{62}$$

$$\|\Phi_h I_h(f)\|_{L^2(D_h)} \leq \|f\|_{L^2} + c h^r |f|_{H^r}. \tag{63}$$

Proof. The first two inequalities were proved in [DH4, Proposition 3.4].

We prove (62). It is sufficient to take $n = 1$. Let

$$u = \Phi(f), \quad u_h = \Phi_h I_h(f).$$

In order to apply an Aubin-Nitsche type argument, define z on D_h by

$$\Delta z = u - u_h \text{ on } D_h, \quad z = 0 \text{ on } \partial D_h.$$

Denoting the pointwise interpolant of z by $I_h z$,

$$\begin{aligned}
\|u - u_h\|^2_{L^2(D_h)} &= \int_{D_h} (u - u_h)\, \Delta z \\
&= \int_{\partial D_h} (u - u_h)\, \frac{\partial z}{\partial \nu} - \int_{D_h} \nabla(u - u_h)\, \nabla z \\
&= \int_{\partial D_h} (u - u_h)\, \frac{\partial z}{\partial \nu} - \int_{D_h} \nabla(u - u_h)\, \nabla(z - I_h z) \\
&\le \|u - u_h\|_{L^2(\partial D_h)}\, \|\nabla z\|_{L^2(\partial D_h)} + |u - u_h|_{H^1(D_h)}\, |z - I_h z|_{H^1(D_h)} \\
&\le \|u - u_h\|_{L^2(\partial D_h)}\, \|z\|_{H^2(D_h)} + ch^{r-1/2} |f|_{H^r}\, h |z|_{H^2(D_h)} \\
&\le \Big(\|u - u_h\|_{L^2(\partial D_h)} + ch^{r+1/2} |f|_{H^r} \Big)\, \|u - u_h\|_{L^2(D_h)},
\end{aligned}$$

by elliptic regularity on convex domains for the last inequality and by a trace theorem, (60) and interpolation for the previous inequality.

Hence

$$\|u - u_h\|_{L^2(D_h)} \le \|u - u_h\|_{L^2(\partial D_h)} + ch^{r+1/2} |f|_{H^r}.$$

But

$$\begin{aligned}
\|u - u_h\|_{L^2(\partial D_h)} &\le c\|u \circ \pi - u_h \circ \pi\|_{L^2} \\
&\le c\big(\|u - u \circ \pi\|_{L^2} + \|u - u_h \circ \pi\|_{L^2} \big) \le ch^2 |f|_{H^1} + \|f - I_h^{\partial D} f\|_{L^2}
\end{aligned}$$

from (72), and since on ∂D one has $u_h \circ \pi = I_h f \circ \pi = I_h^{\partial D} f$.

This gives the result, noting that $|f|_{H^1(\partial D)} \le |f|_{H^{3/2}(\partial D)}$ and using Poincaré's inequality.

Then (63) follows from the triangle inequality as

$$\|\Phi(f)\|_{L^2(D_h)} \le \|\Phi(f)\|_{L^2(D)} \le c\|\Phi(f)\|_{H^{1/2}(D)} \le c\|f\|_{L^2};$$

now apply a standard interpolation inequality to $\|f - I_h^{\partial D} f\|_{L^2}$.

One can estimate the term $\|f - I_h^{\partial D} f\|_{L^2(\partial D)}$ in (62) by $ch^r |f|_{H^r}$, but in some applications it is important to estimate this term by a higher norm in order to keep the coefficient $ch^{r+1/2}$, c.f. the estimate for A in Lemma 4.2. On the other hand, at the end of the proof of (63) we do estimate $\|f - I_h^{\partial D} f\|_{L^2}$ in this manner, as (63) will be applied in conjunction with interpolation or inverse estimates for f, in which case the term $ch^r |f|_{H^r}$ has the natural power of h to balance the term $\|f\|_{L^2}$, c.f. the estimate for B in Lemma 4.2 and in Lemma 4.4.

We will need the inverse estimates (see [DH4, Proposition 5.3]):

$$\begin{aligned}
\|\xi_h\|_{C^0} &\le c|\log h|^{1/2} |\xi_h|_{H^{1/2}} \quad \text{if } \xi_h \in H_h, \\
\|\psi_h\|_{C^0} &\le c(\|\psi_h\|_{L^2} + |\log h|^{1/2} |\psi_h|_{H^1}) \quad \text{if } \psi_h \in X_h.
\end{aligned} \tag{64}$$

We finally note some straightforward estimates from [DH4, Section 3]. If $u : \overline{D} \to I\!R$ is harmonic then

$$\|u\|_{L^2(D\setminus D_h)} \le ch\|u\|_{L^2(\partial D)}, \quad |u|_{H^1(D\setminus D_h)} \le ch|u|_{H^1(\partial D)}. \tag{65}$$

If $f, g : \partial D \to I\!R$ then

$$|fg|_{H^{1/2}} \le \|f\|_{C^0}|g|_{H^{1/2}} + |f|_{H^{1/2}}\|g\|_{C^0}, \tag{66}$$

$$\|fg\|_{H^{1/2}} \le c\|f\|_{C^{0,1}}\|g\|_{H^{1/2}}. \tag{67}$$

If $s_i = id + \sigma_i : \partial D \to S^1$ for $i = 1, 2$ and $g : S^1 \to I\!R$, then

$$\|g \circ s_1\|_{H^{1/2}} \le c\|g\|_{C^1}\|s_1\|_{H^{1/2}} \tag{68}$$

$$|g \circ s_1 - g \circ s_2|_{H^{1/2}} \le c\|g\|_{C^2}(\|s_1\|_{C^{0,1}} + \|s_1 - s_2\|_{C^0})\|s_1 - s_2\|_{H^{1/2}} \tag{69}$$

4.3 Estimates for u and u_h

We need to estimate the difference between the function u corresponding to a pair (s, φ) and the discrete function u_h corresponding to the nearby interpolation pair (s_h, φ_h).

Lemma 4.2. *With u and u_h as in (57) and $c = c(\|\gamma\|_{C^2}, \|s\|_{C^2}, \|\varphi\|_{C^2})$,*

$$\|u\|_{H^1(D)} \le c, \quad \|u_h\|_{H^1(D)} \le c, \quad \|u\|_{H^1(D\setminus D_h)} \le ch,$$
$$\|u - u_h\|_{L^2(D_h)} + h|u - u_h|_{H^1(D_h)} \le ch^2.$$

Proof. The first and second estimates are immediate. The third follows from (65):

$$\begin{aligned}\|u\|_{H^1(D\setminus D_h)} &\le \|u_0\|_{H^1(D\setminus D_h)} + \|\varphi\|_{H^1(D\setminus D_h)}\\ &\le ch\|\gamma \circ s\|_{H^1} + ch \le ch.\end{aligned}$$

For the fourth,

$$\begin{aligned}\|u - u_h\|_{L^2(D_h)} &\le \|\Phi(\gamma \circ s) - \Phi_h I_h(\gamma \circ s_h)\|_{L^2(D_h)} + \|\varphi - \varphi_h\|_{L^2(D_h)}\\ &\le \|\Phi(\gamma \circ s) - \Phi I_h(\gamma \circ s)\|_{L^2(D_h)}\\ &\quad + \|\Phi_h I_h(\gamma \circ s - \gamma \circ s_h)\|_{L^2(D_h)} + ch^2\\ &=: A + B + ch^2.\end{aligned}$$

From (62) with $r = 3/2$,

$$A \le ch^2|\gamma \circ s|_{H^{3/2}} + c\|\gamma \circ s - I_h^{\partial D}\gamma \circ s\|_{L^2} \le ch^2\|\gamma \circ s\|_{H^2} \le ch^2.$$

From (63) with $r = 1$,

$$\begin{aligned}B &\le c\|\gamma \circ s - \gamma \circ s_h\|_{L^2} + ch\|\gamma \circ s - \gamma \circ s_h\|_{H^1}\\ &\le c\|s - s_h\|_{L^2} + ch\|s - s_h\|_{H^1} \le ch^2.\end{aligned}$$

This proves the $L^2(D_h)$ estimate.

The proof of the $H^1(D_h)$ estimate follows in a similar way from (60) and (61), see [DH4, (77)].

We also need the following pointwise bounds.

Lemma 4.3. *For* $c = c(\|\gamma\|_{C^2}, \|s\|_{C^2}, \|\varphi\|_{C^2})$,

$$\|u_h\|_{C^0(D_h)} \le c, \quad \|u_h\|_{C^{0,1}(D_h)} \le c|\log h|^{1/2}.$$

Proof. Writing $I_h u$ for the pointwise interpolant,

$$\begin{aligned}
\|u_h\|_{C^0(D_h)} &\le \|I_h u\|_{C^0(D_h)} + \|I_h u - u_h\|_{C^0(D_h)} \\
&\le \|u\|_{C^0(D)} + c|\log h|^{1/2}\, \|I_h u - u_h\|_{H^1(D_h)} \quad \text{by (64)} \\
&\le c\big(1 + |\log h|^{1/2}(\|I_h u - u\|_{H^1(D_h)} + \|u - u_h\|_{H^1(D_h)})\big) \\
&\le c(1 + |\log h|^{1/2} h) \quad \text{using Lemma 4.2} \\
&\le c.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|u_h|_{C^{0,1}(D_h)} &\le |I_h u|_{C^{0,1}(D_h)} + |I_h u - u_h|_{C^{0,1}(D_h)} \\
&\le |u|_{C^{0,1}(D)} + ch^{-1}\|I_h u - u_h\|_{C^0(D_h)} \\
&\le c|\log h|^{1/2},
\end{aligned}$$

as for the C^0 estimate.

4.4 Estimates for v and v_h

The variation $(\xi_h, \psi_h) \in H_h \times X_{h0}$ is also a variation in the smooth setting, see the remarks following (36). But when interpreted as variations of discrete and smooth functions from D_h and D respectively into $\mathbb{R}^3$, these variations are no longer equal. We need to estimate the difference.

Lemma 4.4. *For* v *and* v_h *in* (58) *and* $c = c(\|\gamma\|_{C^2}, \|s\|_{C^2})$,

$$\|v_0\|_{H^1(D)} \le c|\xi_h|_{H^{1/2}}, \quad \|v_{h0}\|_{H^1(D_h)} \le c|\xi_h|_{H^{1/2}},$$
$$\|v\|_{L^2(D\setminus D_h)} \le ch|\xi_h|_{H^{1/2}}, \quad \|v\|_{H^1(D\setminus D_h)} \le ch^{1/2}|\xi_h|_{H^{1/2}},$$
$$\|v - v_h\|_{L^2(D_h)} \le ch|\xi_h|_{H^{1/2}}.$$

Proof. For the first estimate, using (67),

$$\|v_0\|_{H^1(D)} = \|\Phi(\xi_h\gamma' \circ s)\|_{H^1(D)} \le c\|\xi_h\gamma' \circ s\|_{H^{1/2}} \le c|\xi_h|_{H^{1/2}}.$$

For v_{h0},

$$\begin{aligned}
\|v_{h0}\|_{H^1(D_h)} &= \|\Phi_h I_h(\xi_h\gamma' \circ s_h)\|_{H^1(D_h)} \\
&\le c\|\xi_h\gamma' \circ s_h\|_{H^{1/2}} + ch^{1/2}\|\xi_h\gamma' \circ s_h\|_{H^1} \quad \text{from (61)} \\
&\le c\|\gamma' \circ s_h\|_{C^{0,1}}\left(\|\xi_h\|_{H^{1/2}} + h^{1/2}\|\xi_h\|_{H^1}\right) \quad \text{from (67)} \\
&\le c|\xi_h|_{H^{1/2}},
\end{aligned}$$

using an inverse estimate and noting $\|s_h\|_{C^{0,1}} \le \|s\|_{C^{0,1}} \le c$.

For the third estimate we have from (65)

$$\|v\|_{L^2(D\setminus D_h)} = \|v_0\|_{L^2(D\setminus D_h)} \le ch\|\xi_h\gamma' \circ s_h\|_{L^2} \le ch\|\xi_h\|_{L^2} \le ch|\xi_h|_{H^{1/2}}.$$

The fourth is similar, using an inverse estimate from H^1 to $H^{1/2}$.

Finally,

$$\begin{aligned}\|v-v_h\|_{L^2(D_h)} &= \|\Phi(\xi_h\gamma'\circ s)-\Phi_h I_h(\xi_h\gamma'\circ s_h)\|_{L^2(D_h)}\\ &\le \|\Phi(\xi_h\gamma'\circ s)-\Phi I_h(\xi_h\gamma'\circ s)\|_{L^2(D_h)}\\ &\quad+\|\Phi_h I_h(\xi_h(\gamma'\circ s-\gamma'\circ s_h))\|_{L^2(D_h)}\\ &= A+B.\end{aligned}$$

From (62) with $s=1$,

$$\begin{aligned}A &\le ch^{3/2}|\xi_h\gamma'\circ s|_{H^1}+c\|\xi_h\gamma'\circ s-I_h^{\partial D}(\xi_h\gamma'\circ s)\|_{L^2}\\ &\le ch^{3/2}\|\xi_h\|_{H^1}+ch^2|\xi_h\gamma'\circ s|_{H^2}\quad \text{(piecewise } H^2 \text{ semi-norm)}\\ &\le ch^{3/2}\|\xi_h\|_{H^1}+ch^2\|\xi_h\|_{H^2}\\ &\le ch^{3/2}\|\xi_h\|_{H^1}\quad \text{as } \xi_h''=0 \text{ pointwise}\\ &\le ch|\xi_h|_{H^{1/2}}.\end{aligned}$$

From (63)

$$\begin{aligned}B &\le c\|\xi_h(\gamma'\circ s-\gamma'\circ s_h)\|_{L^2}+ch|\xi_h(\gamma'\circ s-\gamma'\circ s_h)|_{H^1}\\ &\le c\|\gamma'\circ s-\gamma'\circ s_h\|_{C^0}\|\xi_h\|_{L^2}+ch\|\gamma'\circ s-\gamma'\circ s_h\|_{C^{0,1}}\|\xi_h\|_{H^1}\\ &\le c\|s-s_h\|_{C^0}\|\xi_h\|_{L^2}+ch\|s-s_h\|_{C^{0,1}}\|\xi_h\|_{H^1}\\ &\le ch^2(\|\xi_h\|_{L^2}+\|\xi_h\|_{H^1})\quad \text{from (40)}\\ &\le ch^{3/2}|\xi_h|_{H^{1/2}}.\end{aligned}$$

4.5 Estimates for w and w_h

We first note for any $\xi_h\in H_h$:

$$\begin{aligned}\|\xi_h^2\|_{L^2}\le c|\xi_h|^2_{H^{1/2}},\quad |\xi_h^2|_{H^{1/2}}\le c|\log h|^{1/2}|\xi_h|^2_{H^{1/2}},\\ \|\xi_h^2\|_{H^1}\le ch^{-1/2}|\log h|^{1/2}|\xi_h|^2_{H^{1/2}},\quad \|\xi_h^2\|_{H^2}\le ch^{-3/2}|\xi_h|^2_{H^{1/2}}.\end{aligned}\tag{70}$$

The first follows from the Sobolev inequality $\|\xi_h\|_{L^4}\le c|\xi_h|_{H^{1/2}}$. The second from (66) and (64). The third since $\|\xi_h^2\|_{H^1}\le ch^{-1/2}|\xi_h^2|_{H^{1/2}}$. And the fourth since $(\xi_h^2)''=(\xi_h')^2$ and so

$$\|\xi_h^2\|_{H^2}\le c\|\xi_h\|^2_{W^{1,4}}\le ch^{-1/2}\|\xi_h\|^2_{H^1}\le ch^{-3/2}\|\xi_h\|^2_{H^{1/2}},$$

where inverse estimates were used for the second and third inequalities.

Lemma 4.5. *With w and w_h as in* (58), *and $c=c(\|\gamma\|_{C^4},\|s\|_{C^2})$,*

$$\begin{aligned}\|w\|_{L^2(D)}\le c|\xi_h|^2_{H^{1/2}},\quad |w|_{H^1(D)}\le c|\log h|^{1/2}|\xi_h|^2_{H^{1/2}},\\ \|w-w_h\|_{L^2(D_h)}\le ch^{1/2}|\xi_h|^2_{H^{1/2}},\quad |w_h|_{H^1(D_h)}\le c|\log h|^{1/2}|\xi_h|^2_{H^{1/2}},\\ \|w\|_{L^2(D\setminus D_h)}\le ch|\xi_h|_{H^{1/2}}.\end{aligned}$$

Proof. For w we have from (70),

$$\begin{aligned}\|w\|_{L^2(D)}\le \|\xi_h^2\gamma''\circ s\|_{L^2}\le c\|\xi_h^2\|_{L^2}\le c|\xi_h|^2_{H^{1/2}},\\ |w|_{H^1(D)}\le \|\xi_h^2\gamma''\circ s\|_{H^{1/2}}\le c\|\xi_h^2\|_{H^{1/2}}\le c|\log h|^{1/2}|\xi_h|^2_{H^{1/2}}.\end{aligned}$$

For $w - w_h$ we compute

$$\begin{aligned}\|w - w_h\|_{L^2(D_h)} &\le \|\Phi(\xi_h^2\gamma'' \circ s) - \Phi_h I_h(\xi_h^2\gamma'' \circ s_h)\|_{L^2(D_h)} \\ &\quad + \|\Phi_h I_h(\xi_h^2(\gamma'' \circ s - \gamma'' \circ s_h))\|_{L^2(D_h)} \\ &=: A + B.\end{aligned}$$

From (62)

$$\begin{aligned}A &\le ch^{3/2}|\xi_h^2\gamma'' \circ s|_{H^1} + ch^2|\xi_h^2\gamma'' \circ s|_{H^2} \quad \text{(piecewise } H^2 \text{ semi-norm)} \\ &\le ch^{3/2}\|\xi_h^2\|_{H^1} + ch^2\|\xi_h^2\|_{H^2} \\ &\le ch^{1/2}|\xi_h|_{H^{1/2}} \quad \text{from (70)}.\end{aligned}$$

From (63)

$$\begin{aligned}B &\le \|\xi_h^2(\gamma'' \circ s - \gamma'' \circ s_h)\|_{L^2} + ch|\xi_h^2(\gamma'' \circ s - \gamma'' \circ s_h)|_{H^1} \\ &\le \|\gamma'' \circ s - \gamma'' \circ s_h)\|_{C^0}\|\xi_h^2\|_{L^2} + ch\|\gamma'' \circ s - \gamma'' \circ s_h)\|_{C^{0,1}}\|\xi_h^2\|_{H^1} \\ &\le ch^2\|\xi_h^2\|_{H^1} \quad \text{from (40)} \\ &\le ch^{3/2}|\log h|^{1/2}|\xi_h|^2_{H^{1/2}} \quad \text{from (70)}.\end{aligned}$$

This completes the estimate for $w - w_h$.

Next from (61),

$$\begin{aligned}|w_h|_{H^1(D_h)} &\le |\xi_h^2\gamma'' \circ s_h|_{H^{1/2}} + ch^{1/2}|\xi_h^2\gamma'' \circ s_h|_{H^1} \\ &\le c\|\xi_h^2\|_{H^{1/2}} + ch^{1/2}\|\xi_h^2\|_{H^1} \quad \text{from (69)} \\ &\le c|\log h|^{1/2}|\xi_h|^2_{H^{1/2}} \quad \text{from (70)}.\end{aligned}$$

Finally, from (65) and (70),

$$\|w\|_{L^2(D\setminus D_h)} \le ch\|\xi_h^2\gamma'' \circ s_h\|_{L^2} \le ch\|\xi_h^2\|_{L^2} \le ch|\xi_h|^2_{H^{1/2}}.$$

4.6 Estimates near an Interpolant Pair

Again we consider a smooth pair (s, φ) and its interpolant (s_h, φ_h). Consider a (suitably small) discrete variation $(\eta_h, \chi_h) \in H \times H_0^1$. We need to estimate the difference between the discrete functions u_h and $\overline{u}_h$ corresponding to the pairs (s_h, φ_h) and $(s_h + \eta_h, \varphi_h + \chi_h)$ respectively. We also need to estimate the difference between the first variations v_h and $\overline{v}_h$ at u_h and $\overline{u}_h$ respectively in the same direction (ξ_h, ψ_h); similarly for the second variations w_h and $\overline{w}_h$.

In the following Lemma, and in Proposition 5.3 where the Lemma is applied, we impose the restriction $|(\eta_h, \chi_h)|_{H^{1/2}\times H^1} \le L/|\log h|^{1/2}$ for some constant L. Although not necessary, this simplifies the statements and proofs of results, and is the natural restriction (on η_h) in order to give estimates which have at most linear growth in $|\eta_h|_{H^{1/2}}$ and also have the optimal power of $|\log h|$ as a coefficient for small $|\eta_h|_{H^{1/2}}$. In addition, this is no restriction when these results are applied in Theorem 5.5, as they are required there only for $|(\eta_h, \chi_h)|_{H^{1/2}\times H^1} \le ch$ in order to establish the existence and convergence result for discrete stationary points, and for $|(\eta_h, \chi_h)|_{H^{1/2}\times H^1} \le \varepsilon_0/|\log h|$ in order to establish the uniqueness result.

Lemma 4.6. *Suppose $L > 0$ and $|\eta_h|_{H^{1/2}} |\log h|^{1/2} \le L$. Then for some constant $c = c(\|\gamma\|_{C^4}, \|s\|_{C^2}, \|\varphi\|_{C^2}, L)$,*

$$
\begin{gathered}
\|\overline{u}_{h0}\|_{H^1(D_h)} \le c(1 + |\eta_h|_{H^{1/2}}), \quad \|u_{h0} - \overline{u}_{h0}\|_{H^1(D_h)} \le c|\eta_h|_{H^{1/2}}, \\
\|\overline{v}_{h0}\|_{L^2(D_h)} + |\log h|^{-1/2} |\overline{v}_{h0}|_{H^1(D_h)} \le c|\xi_h|_{H^{1/2}}, \\
\|\overline{v}_h - v_h\|_{L^2(D_h)} + |\log h|^{-1/2} |\overline{v}_h - v_h|_{H^1(D_h)} \le c|\eta_h|_{H^{1/2}} |\xi_h|_{H^{1/2}}, \\
\|\overline{w}_h\|_{L^2(D_h)} + |\log h|^{-1/2} |\overline{w}_h|_{H^1(D_h)} \le c|\xi_h|^2_{H^{1/2}}, \\
\|\overline{w}_h - w_h\|_{L^2(D_h)} + |\log h|^{-1/2} |\overline{w}_h - w_h|_{H^1(D_h)} \le \\
\qquad\qquad c|\log h|^{1/2} |\eta_h|_{H^{1/2}} |\xi_h|^2_{H^{1/2}}.
\end{gathered}
$$

If we also assume $|\chi_h|_{H^1} |\log h|^{1/2} \le L$, then $\|\overline{u}_h\|_{C^0(D_h)} \le c$.

Proof. For the $\overline{u}_{h0}$ estimates one has from (63)

$$
\begin{aligned}
&\|u_{h0} - \overline{u}_{h0}\|_{L^2(D_h)} \\
&\quad \le \|\Phi_h I_h(\gamma \circ (s_h + \eta_h) - \gamma \circ s_h)\|_{L^2(D_h)} \\
&\quad \le \|\gamma \circ (s_h + \eta_h) - \gamma \circ \sigma_h\|_{L^2} + ch|\gamma \circ (s_h + \eta_h) - \gamma \circ \sigma_h|_{H^1} \\
&\quad \le c\|\eta_h\|_{L^2} + ch\|\eta_h\|_{H^1} \\
&\quad \le c|\eta_h|_{H^{1/2}}.
\end{aligned}
$$

The estimate for $|u_{h0} - \overline{u}_{h0}|_{H^1(D_h)}$ is similar; using (61) instead of (63) (or see the estimate for A_1 in [DH4, Proposition 5.2]) one first obtains

$$
|u_{h0} - \overline{u}_{h0}|_{H^1(D_h)} \le c(1 + |\log h|^{1/2} |\eta_h|_{H^{1/2}}) |\eta_h|_{H^{1/2}}.
$$

This gives the result. The estimate for $\|\overline{u}_{h0}\|_{H^1(D_h)}$ now follows from that for $\|u_{h0}\|_{H^1(D_h)}$ implicit in Lemma 4.2.

If also $|\chi_h|_{H^1} \le L/|\log h|^{1/2}$ then

$$
\begin{aligned}
\|\overline{u}_h\|_{C^0(D_h)} &\le \|u_h\|_{C^0(D_h)} + \|u_h - \overline{u}_h\|_{C^0(D_h)} \\
&\le c + c|\log h|^{1/2} \|u_h - \overline{u}_h\|_{H^1(D_h)} \quad \text{from Lemma 4.3 and (64)} \\
&\le c + c|\log h|^{1/2} |(\eta_h, \chi_h)|_{H^{1/2} \times H^1} \quad \text{by the second estimate} \\
&\le c \quad \text{by assumption.}
\end{aligned}
$$

We next compute again from (63)

$$
\begin{aligned}
&\|\overline{v}_h - v_h\|_{L^2(D_h)} \\
&\quad = \|\Phi_h I_h(\xi_h(\gamma' \circ (s_h + \eta_h) - \gamma' \circ s_h))\|_{L^2(D_h)} \\
&\quad \le \|\xi_h(\gamma' \circ (s_h + \eta_h) - \gamma' \circ s_h)\|_{L^2} \\
&\qquad\qquad + ch|\xi_h(\gamma' \circ (s_h + \eta_h) - \gamma' \circ s_h)|_{H^1} \\
&\quad \le \|\gamma' \circ (s_h + \eta_h) - \gamma' \circ s_h\|_{L^4} \|\xi_h\|_{L^4} \\
&\qquad\qquad + ch\big(\|\gamma' \circ (s_h + \eta_h) - \gamma' \circ s_h\|_{C^0} |\xi_h|_{H^1} \\
&\qquad\qquad\qquad + |\gamma' \circ (s_h + \eta_h) - \gamma' \circ s_h|_{H^1} \|\xi_h\|_{C^0}\big) \\
&\quad \le c\|\eta_h\|_{L^4} \|\xi_h\|_{L^4} + ch\big(\|\eta_h\|_{C^0} |\xi_h|_{H^1} + \|\eta_h\|_{H^1} \|\xi_h\|_{C^0}\big) \\
&\quad \le c|\eta_h|_{H^{1/2}} |\xi_h|_{H^{1/2}} + ch^{1/2} |\log h|^{1/2} |\eta_h|_{H^{1/2}} |\xi_h|_{H^{1/2}} \quad \text{from (70)} \\
&\quad \le c|\eta_h|_{H^{1/2}} |\xi_h|_{H^{1/2}}.
\end{aligned}
$$

The estimate for $|\overline{v}_h - v_h|_{H^1(D_h)}$ is again similar using (61) (or see the estimates for A_2 in [DH4, Proposition 5.2]). One first obtains

$$|\overline{v}_h|_{H^1(D_h)} \le c\big(|\log h|^{1/2}(1 + |\eta_h|_{H^{1/2}})|\xi_h|_{H^{1/2}} + |\psi_h|_{H^1}\big),$$

which gives the result. The estimate for $\|\overline{v}_h\|_{H^1(D_h)}$ follows from Lemma 4.4.

We finally compute from (63),

$$\begin{aligned}
&\|\overline{w}_h - w_h\|_{L^2(D_h)} \\
&\quad = \|\Phi_h I_h(\xi_h^2(\gamma'' \circ (s_h + \eta_h) - \gamma'' \circ s_h)\|_{L^2(D_h)} \\
&\quad \le \|\xi_h^2(\gamma'' \circ (s_h + \eta_h) - \gamma'' \circ s_h)\|_{L^2} \\
&\qquad\qquad + ch|\xi_h^2(\gamma'' \circ (s_h + \eta_h) - \gamma'' \circ s_h)|_{H^1} \\
&\quad \le c\|\xi_h\|_{L^4}^2 \|\gamma'' \circ (s_h + \eta_h) - \gamma'' \circ s_h\|_{C^0} \\
&\qquad\qquad + ch\Big(\|\xi_h^2\|_{C^0}|\gamma'' \circ (s_h + \eta_h) - \gamma'' \circ s_h|_{H^1} \\
&\qquad\qquad\qquad |\xi_h^2|_{H^1}\|\gamma'' \circ (s_h + \eta_h) - \gamma'' \circ s_h\|_{C^0}\Big) \\
&\quad \le c|\xi_h|_{H^{1/2}}^2 |\eta_h|_{H^{1/2}} |\log h|^{1/2} + ch^{1/2}|\log h|^{1/2}|\xi_h|_{H^{1/2}}^2 \\
&\quad \le c|\log h|^{1/2}|\eta_h|_{H^{1/2}}|\xi_h|_{H^{1/2}}^2.
\end{aligned}$$

Again, the estimate for $|\overline{w}_h - w_h|_{H^1(D_h)}$ follows similarly from (61) (or see the estimate for A_3 in [DH4, Proposition 5.2]). One obtains

$$|\overline{w}_h - w_h|_{H^1(D_h)} \le c|\log h|(1 + |\log h|^{1/2}|\eta_h|_{H^{1/2}})|\eta_h|_{H^{1/2}}|\xi_h|_{H^{1/2}}^2,$$

which yields the result. Lemma 4.5 now gives the estimate for $\|\overline{w}_h\|_{H^1(D_h)}$.

5 Proof of Main Theorem

5.1 Notation

We continue to use the notation of Section 5.1.

By Lemma 4.3 we choose $R = R(\|\gamma\|_{C^2}, \|s\|_{C^2}, \|\varphi\|_{C^2})$ so that

$$\|u\|_{C^0}, \|u_h\|_{C^0} \le R.$$

In Proposition 5.3 we assume $|(\eta_h, \chi_h)|_{H^{1/2}\times H^1} \le L/|\log h|^{1/2}$, and then by Lemma 4.6 we choose $R = R(\|\gamma\|_{C^4}, \|s\|_{C^2}, \|\varphi\|_{C^2}, L)$ so that also

$$\|\overline{u}_h\|_{C^0} \le R.$$

In either case, noting H is C^2, we define

$$M = \sup_{|z|\le R} \{|H(z)|, |H'(z)|, |H''(z)|\}.$$

5.2 First and Second Variation Estimates

We begin by showing that if (s, φ) is stationary then the first variation of the discrete functional at the interpolant pair (s_h, φ_h) is $O(h)$. We next establish non-degeneracy of the discrete energy functional at (s_h, φ_h). Finally we prove an estimate which then implies that the second variation at discrete pairs near (s_h, φ_h) (more precisely, within distance $o(|\log h|^{-3/2})$) is also non-degenerate.

Note that the proof of the following proposition establishes a similar estimate for $|E'(s,\varphi)(\xi_h,\psi_h) - E_h'(s_h,\varphi_h)(\xi_h,\psi_h)|$ if (s,φ) is not stationary.

Proposition 5.1. *Suppose* (s,φ) *is stationary for* E. *Then for* $(\xi_h,\psi_h)\in H_h\times X_{h0}$ *and* $c_1=c_1(\|\gamma\|_{C^3},\|s\|_{C^2},\|\varphi\|_{C^2},M)$, *one has*

$$|E_h'(s_h,\varphi_h)(\xi_h,\psi_h)|\le c_1h|(\xi_h,\psi_h)|_{H^{1/2}\times H^1}.$$

Proof. It follows from the stationarity of (s,φ), (23) and (48), that

$$\begin{aligned}
E_h'(s_h,\varphi_h)(\xi_h,\psi_h) &= \int_{D_h}\nabla u_{h0}\cdot\nabla v_{h0}-\int_D\nabla u_0\cdot\nabla v_0\\
&+\int_{D_h}\nabla\varphi_h\cdot\nabla\psi_h-\int_D\nabla\varphi\cdot\nabla\psi_h\\
&+2\int_{D_h}H(u_h)v_h\cdot u_{hx}\wedge u_{hy}-H(u)v\cdot u_x\wedge u_y\\
&-2\int_{D\setminus D_h}H(u)v\cdot u_x\wedge u_y+\frac{2}{3}\int_{\partial D_h}Q(u_h)\cdot v_h\wedge u_{h\tau}\\
&=: A+B+2C-2D+\frac{2}{3}E.
\end{aligned}$$

One can show by a rearrangement of the terms in A, Lemmas 4.2 and 4.4, and an integration by parts and some boundary estimates to handle the fact that no power of h can be gained from $|v-v_0|_{H^1(D_h)}$, that

$$|A|\le ch|\xi_h|_{H^{1/2}},$$

where c here and subsequently has the same dependencies as in the statement of the proposition. See [DH4, Proposition 4.2] for details.

We now proceed to estimate the new terms arising from the presence of the mean curvature term. By a standard interpolation result,

$$|B|\le ch|\psi_h|_{H^1}.$$

By adding and subtracting terms we can estimate $|C|$ by a sum of terms of the form $c\,a_1a_2a_3a_4$, where

$$\begin{aligned}
a_1 &= \|H(u_h)\|_{C^0(D_h)},\|H(u)\|_{C^0(D_h)}\text{ or }\|u_h-u\|_{L^2(D_h)},\\
a_2 &= \|v_h\|_{L^2(D_h)},\|v\|_{L^2(D_h)}\text{ or }\|v-v_h\|_{L^2(D_h)},\\
a_3,a_4 &= \|\nabla u_h\|_{L^\infty(D_h)},\|\nabla u\|_{C^0(D_h)}\text{ or }\|\nabla u-\nabla u_h\|_{L^2(D_h)},
\end{aligned}$$

and each such term contains exactly one factor which is a difference and at most two L^2 type factors. From Lemmas 4.2, 4.3 and 4.4, it follows

$$|C|\le ch|\xi_h|_{H^{1/2}}.$$

From Lemma 4.4,

$$|D|\le ch|\xi|_{H^{1/2}}.$$

Finally, for E, consider the boundary edge E_j of ∂D_h joining $e^{i\theta_j}$ to $e^{i\theta_{j+1}}$. Let $h_j=|\theta_j-\theta_{j+1}|$ and set $s_j=s(\theta_j)$, $s_{j+1}=s(\theta_{j+1})$, $\xi_j=\xi_h(\theta_j)$ and $\xi_{j+1}=\xi_h(\theta_{j+1})$. By $O(h^\alpha)$ denote any quantity which in absolute value is bounded by ch^α for some c as above.

Then at the point $e^{i\theta_j}+t(e^{i\theta_{j+1}}-e^{i\theta_j})\in\partial D_h$, with $0\le t\le 1$,

$$\begin{aligned}
&|v_h\wedge u_{h\tau}|\\
&\quad=\big|\big((1-t)\xi_j\gamma'(s_j)+t\xi_{j+1}\gamma'(s_{j+1})\big)\wedge\frac{1}{h_j}\big(\gamma(s_{j+1})-\gamma(s_j)\big)\big|
\end{aligned}$$

$$
\begin{aligned}
&= \Big| \big((1-t)\xi_j\gamma'(s_j) + t\xi_{j+1}(\gamma'(s_j)+O(h))\big) \\
&\qquad\qquad \wedge \frac{1}{h_j}\big((s_{j+1}-s_j)\gamma'(s_j)+O(h^2)\big)\Big| \\
&\le ch(|\xi_j|+|\xi_{j+1}|),
\end{aligned}
$$

since $\gamma'(s_j)\wedge\gamma'(s_j)=0$. In particular,

$$
\|v_h\wedge u_{h\tau}\|_{L^2(\partial D_h)} \le ch\|\xi_h\|_{L^2} \le ch|\xi_h|_{H^{1/2}}. \tag{71}
$$

and so

$$
|E| \le ch|\xi|_{H^{1/2}}.
$$

This completes the proof.

Proposition 5.2. *Suppose $E''(s,\varphi)$ is non-degenerate with non-degeneracy constant $\lambda>0$. Then $E_h''(s_h,\varphi_h)$ is non-degenerate with non-degeneracy constant $\ge 3\lambda/4$ for all $h\le h_1 = h_1(\|\gamma\|_{C^4},\|s\|_{C^2},\|\varphi\|_{C^2},M,\nu)$, with ν as in* (33).

Proof. Suppose $(\xi_h,\psi_h)\in H_h\times X_{h0}$. From (22), (24), (47) and (49),

$$
\begin{aligned}
&\big(E''(s,\varphi)-E_h''(s_h,\varphi_h)\big)(\xi_h,\psi_h)^2 \\
&= \Big(\int_D |\nabla v_0|^2 + \int_D \nabla u_0\cdot\nabla w - \int_{D_h}|\nabla v_{h0}|^2 - \int_{D_h}\nabla u_{h0}\cdot\nabla w_h\Big) \\
&\quad + \ 2\int_{D\setminus D_h} H(u)\big(w\cdot u_x\wedge u_y + v\cdot(v_x\wedge u_y+u_x\wedge v_y)\big) \\
&\quad + \ 2\int_{D\setminus D_h} H'(u)\cdot v\ v\cdot u_x\wedge u_y \\
&\quad + \ 2\int_{D_h} H(u)w\cdot u_x\wedge u_y - H(u_h)w_h\cdot u_{hx}\wedge u_{hy} \\
&\quad + \ 2\int_{D_h} H(u)v\cdot(v_x\wedge u_y+u_x\wedge v_y) \\
&\qquad - \ 2\int_{D_h} H(u_h)v_h\cdot(v_{hx}\wedge u_{hy}+u_{hx}\wedge v_{hy}) \\
&\quad + \ 2\int_{D_h} H'(u)\cdot v\ v\cdot u_x\wedge u_y - H'(u_h)\cdot v_h\ v_h\cdot u_{hx}\wedge u_{hy} \\
&\quad - \ \frac{2}{3}\int_{\partial D_h} Q(u_h)\cdot(w_h\wedge u_{h\tau}+v_h\wedge v_{h\tau}) \\
&\quad - \ \frac{2}{3}\int_{\partial D_h} (Q'(u_h)v_h)\cdot v_h\wedge u_{h\tau} \\
&=: A+2B+2C+2D+2E+2F-\frac{2}{3}G-\frac{2}{3}H,
\end{aligned}
$$

where the subscript τ denotes differentiation along ∂D_h and $Q'(u_h)v_h$ is interpreted as matrix-vector multiplication.

In [DH4, Proposition 4.3] we showed that

$$
A = -|v_0-v_{h0}|^2_{H^1(D_h)} + R_0, \quad |R_0|\le ch^{1/2}|\log h|^{1/2}|\xi_h|^2_{H^{1/2}},
$$

where $c=c(\|\gamma\|_{C^3},\|s\|_{C^2})$.

From Lemmas 4.3, 4.4 and 4.5,

$$|B| \le c\big(h\|w\|_{L^2(D\setminus D_h)} + \|v\|^2_{H^1(D\setminus D_h)}\big) \le ch^2|\xi_h|^2_{H^{1/2}}.$$

Similarly,

$$|C| \le c\|v\|^2_{L^2(D\setminus D_h)} \le ch^2|\xi_h|^2_{H^{1/2}}.$$

By adding and subtracting terms we can estimate $|D|$ by a sum of terms of the form $c\,a_1a_2a_3a_4$, where

$$\begin{aligned} a_1 &= \|H(u)\|_{C^0(D_h)}, \|H(u_h)\|_{C^0(D_h)} \text{ or } \|u_h - u\|_{L^2(D_h)}, \\ a_2 &= \|w\|_{L^2(D_h)}, \|w_h\|_{L^2(D_h)} \text{ or } \|w - w_h\|_{L^2(D_h)}, \\ a_3, a_4 &= \|\nabla u\|_{C^0(D_h)}, \|\nabla u_h\|_{L^\infty(D_h)} \text{ or } \|\nabla u - \nabla u_h\|_{L^2(D_h)}, \end{aligned}$$

and each such term contains exactly one factor which is a difference and at most two L^2 type factors. From Lemmas 4.2 and 4.5 it follows that

$$|D| \le ch(|\xi_h|^2_{H^{1/2}} + |\psi_h|^2_{H^1}),$$

where here and until further notice $c = c(\|\gamma\|_{C^3}, \|s\|_{C^2}, \|\varphi\|_{C^2}, M)$.

The difficulty in estimating E is that we gain no power of h from the term $|v - v_h|_{H^1(D_h)}$. Setting

$$E^* = \int_{D_h} H(u)v \cdot (v_x \wedge u_y + u_x \wedge v_y - v_{hx} \wedge u_{hy} - u_{hx} \wedge v_{hy}),$$

and adding and subtracting terms we have

$$\begin{aligned} |E| &\le c\|u - u_h\|_{L^4(D_h)}\|v\|_{L^4(D_h)}\|\nabla v\|_{L^2(D_h)} \\ &\qquad + c\|v - v_h\|_{L^2(D_h)}\|\nabla v\|_{L^2(D_h)} + |E^*| \\ &\le c\|u - u_h\|_{H^1(D_h)}\|v\|^2_{H^1(D_h)} \\ &\qquad + c\|v - v_h\|_{L^2(D_h)}\|\nabla v\|_{L^2(D_h)} + |E^*| \\ &\le ch|\xi_h|^2_{H^{1/2}} + |E^*|. \end{aligned}$$

For E^*, integration by parts gives

$$\begin{aligned} E^* &= \int_{D_h} v \cdot \big((H(u)v)_x \wedge u_y + u_x \wedge (H(u)v)_y\big) \\ &\quad - \int_{D_h} v_h \cdot \big((H(u)v)_x \wedge u_{hy} + u_{hx} \wedge (H(u)v)_y\big) \\ &\quad - \int_{\partial D_h} H(u)v \cdot v_h \wedge u_{h\tau}. \end{aligned}$$

(The integration by parts is valid for smooth functions, and hence for piecewise linear functions by approximation.) The difference of the first two integrals is bounded in absolute value by $ch|\log h|^{1/2}\,(|\xi_h|^2_{H^{1/2}} + |\psi_h|^2_{H^1})$. This comes from adding and subtracting terms as usual and using Lemmas 4.2 and 4.4 and the inverse estimate $\|v_h\|_{C^0(D_h)} \le |\log h|^{1/2}\|v_h\|_{H^1(D_h)}$. From (71) we have that $\|v_h \wedge u_{h\tau}\|_{L^2(\partial D_h)} \le ch|\xi_h|_{H^{1/2}}$. On the other hand

$$\|v\|_{L^2(D_h)} \le c\|v\|_{H^1(D_h)} \le c(|\xi_h|_{H^{1/2}} + |\psi_h|_{H^1}),$$

and so the boundary integral is bounded by $ch(|\xi_h|^2_{H^{1/2}} + |\psi_h|^2_{H^1})$. Hence

$$|E^*| \le ch|\log h|^{1/2}\,(|\xi_h|^2_{H^{1/2}} + |\psi_h|^2_{H^1}).$$

It follows that
$$|E| \le ch|\log h|^{1/2}\,(|\xi_h|^2_{H^{1/2}} + |\psi_h|^2_{H^1}).$$

The estimate for F follows in the usual way from Lemmas 4.2 and 4.4, giving
$$|F| \le ch(|\xi_h|^2_{H^{1/2}} + |\psi_h|^2_{H^1}).$$

The estimate for G is by a similar idea to that for E in the previous proposition. Let E_j be the boundary edge of ∂D_h joining $e^{i\theta_j}$ to $e^{i\theta_{j+1}}$. Let $h_j = |\theta_j - \theta_{j+1}|$ and set $s_j = s(\theta_j)$, $s_{j+1} = s(\theta_{j+1})$, $\xi_j = \xi_h(\theta_j)$ and $\xi_{j+1} = \xi_h(\theta_{j+1})$.

At the point $e^{i\theta_j} + t(e^{i\theta_{j+1}} - e^{i\theta_j})$ on the edge E_j,
$$\begin{aligned}
&w_h \wedge u_{h\tau} + v_h \wedge v_{h\tau}\\
&\quad= \Big((1-t)\xi_j^2\gamma''(s_j) + t\xi_{j+1}^2\gamma''(s_{j+1})\Big) \wedge \frac{1}{h_j}\Big(\gamma(s_{j+1}) - \gamma(s_j)\Big)\\
&\qquad + \frac{1}{h_j}\xi_j\xi_{j+1}\gamma'(s_j) \wedge \gamma'(s_{j+1})\\
&\quad= \Big((1-t)\xi_j^2\gamma''(s_j) + t\xi_{j+1}^2\big(\gamma''(s_j) + O(h)\big)\Big)\\
&\qquad \wedge \frac{1}{h_j}\Big((s_{j+1}-s_j)\gamma'(s_j) + O(h^2)\Big)\\
&\qquad + \frac{1}{h_j}\xi_j\xi_{j+1}\Big(\gamma'(s_j) \wedge (s_{j+1}-s_j)\gamma''(s_j) + O(h^2)\Big)\\
&\quad= \frac{s_{j+1}-s_j}{h_j}\Big(\xi_j\xi_{j+1} - \xi_j^2 + t\xi_j^2 - t\xi_{j+1}^2\Big)\gamma'(s_j) \wedge \gamma''(s_j)\\
&\qquad + O(h)\big(\xi_j^2 + \xi_{j+1}^2\big).
\end{aligned}$$
Integrating with respect to t from 0 to 1, and noting $|s_{j+1} - s_j|/h_j \le c$,
$$\begin{aligned}
|G| &\le c\sum_j h_j\Big((\xi_{j+1}-\xi_j)^2 + O(h)(\xi_{j+1}^2 + \xi_j^2)\Big)\\
&\le ch^2|\xi_h|^2_{H^1} + O(h)\|\xi_h\|^2_{L^2} \le ch|\xi_h|^2_{H^{1/2}}.
\end{aligned}$$

Finally, for H note that on E_j, $|v_h| \le c(|\xi_j| + |\xi_{j+1}|)$, hence
$$\|v_h\|_{L^2(\partial D_h)} \le c\|\xi_h\|_{L^2(\partial D)} \le c|\xi_h|_{H^{1/2}}.$$
Together with (71) this implies
$$|H| \le ch|\xi_h|^2_{H^{1/2}}.$$

Putting together the estimates for B–H, it follows that
$$\begin{aligned}
&E_h''(s_h,\varphi_h)(\xi_h,\psi_h)^2 = E''(s,\varphi)(\xi_h,\psi_h)^2 + |v - v_h|^2_{H^1(D_h)} + R\\
&R \le ch^{1/2}|\log h|^{1/2}(|\xi_h|^2_{H^{1/2}} + |\psi_h|^2_{H^1}).
\end{aligned} \tag{72}$$

If (s,φ) is a strictly stable stationary point, this already implies the stated result (with h_0 independent of ν).

For future reference we note from (72), (25) and Lemma 4.4, or it can be checked directly, that
$$\begin{aligned}
&|E_h''(s_h,\varphi_h)(\xi_h,\psi_h)^2| \le c(|\xi_h|^2_{H^{1/2}} + |\psi_h|^2_{H^1})\\
&|E_h''(s_h,\varphi_h)\big((\xi_h,\psi_h),(\eta_h,\chi_h)\big)| \le
\end{aligned} \tag{73}$$

$$c\big(|\xi_h|_{H^{1/2}} + |\psi_h|_{H^1}\big)\big(|\eta_h|_{H^{1/2}} + |\chi_h|_{H^1}\big).$$

If $E''(s,\varphi)$ has non-trivial negative eigenspace, then we compute

$$\begin{aligned}
&E_h''(s_h,\varphi_h)(\xi_h,\psi_h)\big((\xi_h,\psi_h)^{(+)} - (\xi_h,\psi_h)^{(+)}\big)\\
&\quad = E_h''(s_h,\varphi_h)\big((\xi_h,\psi_h)^{(+)}\big)^2 - E_h''(s_h,\varphi_h)\big((\xi_h,\psi_h)^{(-)}\big)^2 \quad \text{by (53)}\\
&\quad \geq \Big(E''(s,\varphi)\big((\xi_h,\psi_h)^{(+)}\big)^2 - E''(s,\varphi)\big((\xi_h,\psi_h)^{(-)}\big)^2\Big)\\
&\qquad - |v^{(-)} - v_h^{(-)}|^2_{H^1(D_h)} - ch^{1/2}|\log h|^{1/2}(|\xi_h|^2_{H^{1/2}} + |\psi|^2_{H^1}) \text{ by (72)}\\
&\quad =: A - B - C,
\end{aligned}$$

where

$$\begin{gathered}
v^{(-)} = \Phi(\xi_h^{(-)}\gamma'\circ s), \quad v_h^{(-)} = \Phi_h I_h(\xi_h^{(-)}\gamma'\circ s_h),\\
(\xi_h^{(-)},\psi_h^{(-)}) = (\xi_h,\psi_h)^{(-)}.
\end{gathered}$$

But

$$\begin{aligned}
A &= E''(s,\varphi)(\xi_h,\psi_h)\big((\xi_h,\psi_h)^{(+)} - (\xi_h,\psi_h)^{(-)}\big) \quad \text{by (53)}\\
&= E''(s,\varphi)(\xi_h,\psi_h)\big((\xi_h,\psi_h)^{+} - (\xi_h,\psi_h)^{-}\big) +\\
&\quad E''(s,\varphi)(\xi_h,\psi_h)\big((\xi_h,\psi_h)^{(+)} - (\xi_h,\psi_h)^{+} + (\xi_h,\psi_h)^{(-)} - (\xi_h,\psi_h)^{-}\big)\\
&\geq (\lambda - ch)(|\xi_h|^2_{H^{1/2}} + |\psi_h|^2_{H^1}),
\end{aligned}$$

from (31), (73) and (54), where now and for the remainder of the proof,
$c = c(\|\gamma\|_{C^3}, \|s\|_{C^2}, \|\varphi\|_{C^2}, M, \nu)$.

Also,

$$\begin{aligned}
B^{1/2} &\leq \Big|\Phi(\xi_h^{(-)}\gamma'\circ s) - \Phi_h I_h(\xi_h^{(-)}\gamma'\circ s)\big|_{H^1(D_h)}\\
&\quad + \big|\Phi_h I_h\big(\xi_h^{(-)}(\gamma'\circ s - \gamma'\circ s_h)\big)\big|_{H^1(D_h)}\\
&\leq ch^{1/2}|\xi_h^{(-)}\gamma'\circ s)|_{H^1}|\xi_h^{(-)}(\gamma'\circ s - \gamma'\circ s_h)\big|_{H^{1/2}}\\
&\quad + ch^{1/2}|\xi_h^{(-)}(\gamma'\circ s - \gamma'\circ s_h)\big|_{H^1} \quad \text{from (60) and (61)}\\
&\leq ch^{1/2}|\xi_h^{(-)}|_{H^1}\\
&\qquad + c\|\gamma'\circ s - \gamma'\circ s_h\|_{C^{0,1}}\big(|\xi_h^{(-)}|_{H^{1/2}} + h^{1/2}|\xi_h^{(-)}|_{H^1}\big) \text{ by (67)}\\
&\leq ch^{1/2}|\xi_h^{(-)}|_{H^{1/2}} \quad \text{from (33) and since } \|s - s_h\|_{C^{0,1}} \leq ch\\
&\leq ch^{1/2}(|\xi_h|_{H^{1/2}} + |\psi_h|_{H^1}) \quad \text{from (55).}
\end{aligned}$$

It follows that

$$E_h''(s_h,\varphi_h)(\xi_h,\psi_h)\big((\xi_h,\psi_h)^{(+)} - (\xi_h,\psi_h)^{(+)}\big) > \frac{3\lambda}{4}(|\xi_h|^2_{H^{1/2}} + |\psi_h|^2_{H^1})$$

for $h \leq \overline{h}_1 = \overline{h}_1(\|\gamma\|_{C^3}, \|s\|_{C^2}, \|\varphi\|_{C^2}, M, \nu)$. Finally,

$$|(\xi_h,\psi_h)^{(+)} - (\xi_h,\psi_h)^{(-)}|^2_{H^{1/2}\times H^1} \leq (1 + ch^2)|(\xi_h,\psi_h)|^2_{H^{1/2}\times H^1},$$

from (54) and the orthogonal decomposition (30). Hence $E_h''(s_h,\varphi_h)$ is non-degenerate by (51), with non-degeneracy constant $\geq 3\lambda/4$, provided $h \leq h_1$ where
$h_1 = h_1(\|\gamma\|_{C^3}, \|s\|_{C^2}, \|\varphi\|_{C^2}, M, \nu)$.

Proposition 5.3. *Suppose $L > 0$ and $|(\eta_h, \chi_h)|_{H^{1/2}\times H^1} \le L/|\log h|^{1/2}$. Then for $(\xi_h, \psi_h) \in H \times H^1_0$ and $c_2 = c_2\|\gamma\|_{C^4}, \|s\|_{C^2}, \|\varphi\|_{C^2}, L, M)$, one has*

$$\Big|\big(E''_h(s_h,\varphi_h) - E''_h(s_h+\eta_h,\varphi_h+\chi_h)\big)(\xi_h,\psi_h)^2\Big|$$
$$\le c_2|\log h|^{3/2}|(\eta_h,\chi_h)|_{H^{1/2}\times H^1}|(\xi_h,\psi_h)|^2_{H^{1/2}\times H^1}.$$

Proof. From (47) and (49)

$$\begin{aligned}
&\big(E''_h(s_h,\varphi_h) - E''_h(s_h+\eta_h,\varphi_h+\chi_h)\big)(\xi_h,\psi_h)^2\\
&= \int_{D_h} |\nabla v_{h0}|^2 + \nabla u_{h0}\cdot\nabla w_h - |\nabla\overline{v}_{h0}|^2 - \nabla\overline{u}_{h0}\cdot\nabla\overline{w}_h\\
&\quad + \quad 2\int_{D_h} H(u_h)w_h\cdot u_{hx}\wedge u_{hy} - H(\overline{u}_h)\overline{w}_h\cdot\overline{u}_{hx}\wedge\overline{u}_{hy}\\
&\quad + \quad 2\int_{D_h} H(u_h)v_h\cdot(v_{hx}\wedge u_{hy} + u_{hx}\wedge v_{hy})\\
&\qquad\qquad - 2\int_{D_h} H(\overline{u}_h)\overline{v}_h\cdot(\overline{v}_{hx}\wedge\overline{u}_{hy} + \overline{u}_{hx}\wedge\overline{v}_{hy})\\
&\quad + \quad 2\int_{D_h} H'(u_h)\cdot v_h\ v_h\cdot u_{hx}\wedge u_{hy} - H'(\overline{u}_h)\cdot\overline{v}_h\ \overline{v}_h\cdot\overline{u}_{hx}\wedge\overline{u}_{hy}\\
&\quad - \quad \frac{2}{3}\int_{\partial D_h} Q(u_h)\cdot(w_h\wedge u_{h\tau} + v_h\wedge v_{h\tau})\\
&\qquad\qquad + \frac{2}{3}\int_{\partial D_h} Q(\overline{u}_h)\cdot(\overline{w}_h\wedge\overline{u}_{h\tau} + \overline{v}_h\wedge\overline{v}_{h\tau})\\
&\quad - \quad \frac{2}{3}\int_{\partial D_h} (Q'(u_h)v_h)\cdot v_h\wedge u_{h\tau} - (Q'(\overline{u}_h)\overline{v}_h)\cdot\overline{v}_h\wedge\overline{u}_{h\tau}\\
&=: A + 2B + 2C + 2D - \frac{2}{3}E - \frac{2}{3}F.
\end{aligned}$$

From Lemmas (4.2)–(4.6) (or [DH4, Proposition 5.2]) we estimate A by

$$\begin{aligned}
|A| &\le \|\nabla(v_h - \overline{v}_h)\|_{L^2(D_h)}\big(\|\nabla v_h\|_{L^2(D_h)} + \|\nabla\overline{v}_h\|_{L^2(D_h)}\big)\\
&\quad + \|\nabla(u_h - \overline{u}_h)\|_{L^2(D_h)}\|\nabla w_h\|_{L^2(D_h)}\\
&\quad + \|\nabla(w_h - \overline{w}_h)\|_{L^2(D_h)}\|\nabla\overline{u}_h\|_{L^2(D_h)}\\
&\le c|\log h|\,|(\eta_h,\chi_h)|_{H^{1/2}\times H^1}\,|(\xi_h,\psi_h)|^2_{H^{1/2}\times H^1},
\end{aligned}$$

From Lemmas 4.2–4.6, the inverse estimate (64) to obtain

$$\|\overline{w}_h\|_{C^0(D_h)} \le c|\log h|^{1/2}\|\overline{w}_h\|_{H^1(D_h)},$$

and noting $|\overline{u}_h|_{H^1(D_h)}$ is bounded from Lemma 4.6 and the assumption on $|(\eta_h,\chi_h)|_{H^{1/2}\times H^1}$, we obtain the following. The highest power of $|\log h|$ comes from the second term.

$$\begin{aligned}
|B| &= \Big|\int_{D_h} (H(u_h) - H(\overline{u}_h))w_h\cdot u_{hx}\wedge u_{hy}\\
&\quad + H(\overline{u}_h)(w_h - \overline{w}_h)\cdot u_{hx}\wedge u_{hy}\\
&\quad + H(\overline{u}_h)\overline{w}_h\cdot(u_{hx} - \overline{u}_{hx})\wedge u_{hy} + H(\overline{u}_h)\overline{w}_h\cdot\overline{u}_{hx}\wedge(u_{hy} - \overline{u}_{hy})\Big|\\
&\le c\|u_h - \overline{u}_h\|_{L^2(D_h)}\|w_h\|_{L^2(D_h)}|u_h|^2_{C^{0,1}(D_h)}
\end{aligned}$$

$$
\begin{aligned}
&+ c\|w_h - \overline{w}_h\|_{L^2(D_h)}|u_h|^2_{C^{0,1}(D_h)}\\
&+ c\|\overline{w}_h\|_{L^2(D_h)}|u_h - \overline{u}_h|_{H^1(D_h)}|u_h|_{C^{0,1}(D_h)}\\
&+ c\|\overline{w}_h\|_{C^0(D_h)}|\overline{u}_h|_{H^1(D_h)}|u_h - \overline{u}_h|_{H^1(D_h)}\\
\le\ & c|\log h|^{3/2}|(\eta_h,\chi_h)|_{H^{1/2}\times H^1}|\xi_h|^2_{H^{1/2}},
\end{aligned}
$$

Similarly, using the same Lemmas and (64),

$$
\begin{aligned}
|C| = \Bigg| \int_{D_h} & (H(u_h) - H(\overline{u}_h))v_h \cdot v_{hx} \wedge u_{hy} + H(\overline{u}_h)(v_h - \overline{v}_h)\cdot v_{hx} \wedge u_{hy}\\
&+ H(\overline{u}_h)\overline{v}_h \cdot (v_{hx} - \overline{v}_{hx}) \wedge u_{hy} + H(\overline{u}_h)\overline{v}_h \cdot \overline{v}_{hx} \wedge (u_{hy} - \overline{u}_{hy})\\
&+ \text{(four similar terms)} \Bigg|\\
\le\ & c\|u_h - \overline{u}_h\|_{L^2(D_h)}\|v_h\|_{C^0(D_h)}|v_h|_{H^1(D_h)}|u_h|_{C^{0,1}(D_h)}\\
&+ c\|v_h - \overline{v}_h\|_{L^2(D_h)}|v_h|_{H^1(D_h)}|u_h|_{C^{0,1}(D_h)}\\
&+ c\|\overline{v}_h\|_{L^2(D_h)}|v_h - \overline{v}_h|_{H^1(D_h)}|u_h|_{C^{0,1}(D_h)}\\
&+ c\|\overline{v}_h\|_{C^0(D_h)}|\overline{v}_h|_{H^1(D_h)}|u_h - \overline{u}_h|_{H^1(D_h)}\\
\le\ & c|\log h|^{3/2}|(\eta_h,\chi_h)|_{H^{1/2}\times H^1}|(\xi_h,\psi_h)|^2_{H^{1/2}\times H^1}.
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
|D| = \Bigg| \int_{D_h} & (H'(u_h) - H'(\overline{u}_h))\cdot v_h\ v_h \cdot u_{hx} \wedge u_{hy}\\
&+ H'(\overline{u}_h)\cdot (v_h - \overline{v}_h)\ v_h \cdot u_{hx} \wedge u_{hy}\\
&+ H'(\overline{u}_h)\cdot \overline{v}_h\ (v_h - \overline{v}_h)\cdot u_{hx} \wedge u_{hy}\\
&+ H'(\overline{u}_h)\cdot \overline{v}_h\ \overline{v}_h \cdot (u_{hx} - \overline{u}_{hx}) \wedge u_{hy}\\
&+ H'(\overline{u}_h)\cdot \overline{v}_h\ \overline{v}_h \cdot \overline{u}_{hx} \wedge (u_{hx} - \overline{u}_{hx}) \Bigg|\\
\le\ & c\|u_h - \overline{u}_h\|_{L^2(D_h)}\|v_h\|_{L^2(D_h)}\|v_h\|_{C^0(D_h)}|u_h|^2_{C^{0,1}(D_h)}\\
&+ c\|v_h - \overline{v}_h\|_{L^2(D_h)}\|v_h\|_{L^2(D_h)}|u_h|^2_{C^{0,1}(D_h)}\\
&+ c\|\overline{v}_h\|_{L^2(D_h)}\|v_h - \overline{v}_h\|_{L^2(D_h)}|u_h|^2_{C^{0,1}(D_h)}\\
&+ c\|\overline{v}_h\|_{L^2(D_h)}\|\overline{v}_h\|_{C^0(D_h)}|u_h - \overline{u}_h|_{H^1(D_h)}|u_h|_{C^{0,1}(D_h)}\\
&+ c\|\overline{v}_h\|^2_{C^0(D_h)}|\overline{u}_h|_{H^1(D_h)}|u_h - \overline{u}_h|_{H^1(D_h)}\\
\le\ & c|\log h|^{3/2}|(\eta_h,\chi_h)|_{H^{1/2}\times H^1}|(\xi_h,\psi_h)|^2_{H^{1/2}\times H^1}.
\end{aligned}
$$

We have

$$
\begin{aligned}
E = \int_{\partial D_h} & (Q(u_h) - Q(\overline{u}_h))\cdot (w_h \wedge u_{h\tau} + v_h \wedge v_{h\tau})\\
&+ Q(\overline{u}_h)\cdot \big((w_h \wedge u_{h\tau} + v_h \wedge v_{h\tau}) - (\overline{w}_h \wedge \overline{u}_{h\tau} + \overline{v}_h \wedge \overline{v}_{h\tau})\big)\\
=:\ & E_1 + E_2.
\end{aligned}
$$

As for G in the previous proposition, using Lemma 4.6 and (64)

$$
\begin{aligned}
|E_1| &\le c\|u_h - \overline{u}_h\|_{C^0(D_h)} h |\xi_h|^2_{H^{1/2}}\\
&\le ch|\log h|^{1/2}|(\eta_h,\chi_h)|_{H^{1/2}\times H^1}|\xi_h|^2_{H^{1/2}}.
\end{aligned}
$$

To estimate E_2, let E_j be the boundary edge of ∂D_h joining $e^{i\theta_j}$ to $e^{i\theta_{j+1}}$. Let $h_j = |\theta_j - \theta_{j+1}|$ and set $s_j = s_h(\theta_j)$, $s_{j+1} = s_h(\theta_{j+1})$, $\eta_j = \eta_h(\theta_j)$, $\eta_{j+1} = \eta_h(\theta_{j+1})$, $\xi_j = \xi_h(\theta_j)$ and $\xi_{j+1} = \xi_h(\theta_{j+1})$. We first compute

$$
\begin{aligned}
v_h \wedge v_{h\tau} - \overline{v}_h \wedge \overline{v}_{h\tau}
&= \frac{1}{h_j}\xi_j\xi_{j+1}\Big(\gamma'(s_j)\wedge\gamma'(s_{j+1}) - \gamma'(s_j+\eta_j)\wedge\gamma'(s_{j+1}+\eta_{j+1})\Big) \\
&= \frac{1}{h_j}\xi_j\xi_{j+1}\Big(\gamma'(s_j)\wedge(s_{j+1}-s_j)\int_0^1 \gamma''(s_j + r(s_{j+1}-s_j))\,dr \\
&\qquad\qquad - \gamma'(s_j+\eta_j)\wedge(s_{j+1}+\eta_{j+1}-s_j-\eta_j) \\
&\qquad\qquad\qquad \int_0^1 \gamma''\big(s_j+\eta_j+r(s_{j+1}+\eta_{j+1}-s_j-\eta_j)\big)\,dr\Big) \\
&= \mathcal{E}_1 + \mathcal{E}_2,
\end{aligned}
$$

where (using $|s_{j+1}-s_j|/h_j \le |s|_{C^1} \le c$)

$$
\begin{aligned}
\mathcal{E}_1 &= -\xi_j\xi_{j+1}\frac{\eta_{j+1}-\eta_j}{h_j}\gamma'(s_j+\eta_j) \\
&\qquad\qquad \wedge \int_0^1 \gamma''\big(s_j+\eta_j+r(s_{j+1}+\eta_{j+1}-s_j-\eta_j)\big)\,dr, \\
|\mathcal{E}_2| &\le c|\xi_j|\,|\xi_{j+1}|\,(|\eta_j|+|\eta_{j+1}|).
\end{aligned}
$$

We similarly compute

$$
\begin{aligned}
w_h \wedge u_{h\tau} - \overline{w}_h \wedge \overline{u}_{h\tau}
&= -\frac{1}{h_j}\big(\gamma(s_{j+1})-\gamma(s_j)\big)\wedge\big((1-t)\xi_j^2\gamma''(s_j)+t\xi_{j+1}^2\gamma''(s_{j+1})\big) \\
&\quad + \frac{1}{h_j}\big(\gamma(s_{j+1}+\eta_{j+1})-\gamma(s_j+\eta_j)\big) \\
&\qquad\qquad \wedge\big((1-t)\xi_j^2\gamma''(s_j+\eta_j)+t\xi_{j+1}^2\gamma''(s_{j+1}+\eta_{j+1})\big) \\
&= -\frac{1}{h_j}(s_{j+1}-s_j)\int_0^1\gamma'(s_j+r(s_{j+1}-s_j))\,dr \\
&\qquad\qquad \wedge\big((1-t)\xi_j^2\gamma''(s_j)+t\xi_{j+1}^2\gamma''(s_{j+1})\big) \\
&\quad + \frac{1}{h_j}(s_{j+1}+\eta_{j+1}-s_j-\eta_j) \\
&\qquad\quad \int_0^1\gamma'\big(s_j+\eta_j+r(s_{j+1}+\eta_{j+1}-s_j-\eta_j)\big)\,dr \\
&\qquad\qquad \wedge\big((1-t)\xi_j^2\gamma''(s_j+\eta_j)+t\xi_{j+1}^2\gamma''(s_{j+1}+\eta_{j+1})\big) \\
&= \mathcal{E}_3 + \mathcal{E}_4,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{E}_3 &= \frac{1}{h_j}(\eta_{j+1}-\eta_j)\int_0^1\gamma'\big(s_j+\eta_j+r(s_{j+1}+\eta_{j+1}-s_j-\eta_j)\big)\,dr \\
&\qquad\qquad \wedge\big((1-t)\xi_j^2\gamma''(s_j+\eta_j)+t\xi_{j+1}^2\gamma''(s_{j+1}+\eta_{j+1})\big), \\
|\mathcal{E}_4| &\le c(|\xi_j|^2+|\xi_{j+1}|^2)\,(|\eta_j|+|\eta_{j+1}|).
\end{aligned}
$$

In order to estimate $\mathcal{E}_1 + \mathcal{E}_3$ pointwise on the edge E_j, consider

$$\begin{aligned}
&\Big|(1-t)\mathcal{E}_1 + \frac{1}{h_j}(\eta_{j+1}-\eta_j)\int_0^1 \gamma'\big(s_j+\eta_j+r(s_{j+1}+\eta_{j+1}-s_j-\eta_j)\big)\,dr \\
&\qquad\qquad\qquad\qquad\qquad \wedge (1-t)\xi_j^2\gamma''(s_j+\eta_j)\Big| \\
&= \Big|\ (1-t)\frac{\eta_{j+1}-\eta_j}{h_j}\ \Big(-\xi_j\xi_{j+1}\gamma'(s_j+\eta_j) \\
&\qquad\qquad \wedge\ \int_0^1 \gamma''\big(s_j+\eta_j+r(s_{j+1}+\eta_{j+1}-s_j-\eta_j)\big)\,dr \\
&\qquad + \xi_j^2\int_0^1 \gamma'\big(s_j+\eta_j+r(s_{j+1}+\eta_{j+1}-s_j-\eta_j)\big)\,dr \\
&\qquad\qquad \wedge\ \gamma''(s_j+\eta_j)\Big)\Big| \\
&= \Big|\ (1-t)\frac{\eta_{j+1}-\eta_j}{h_j} \\
&\qquad \Big((-\xi_j\xi_{j+1}+\xi_j^2)\gamma'(s_j+\eta_j)\wedge\gamma''(s_j+\eta_j) \\
&\qquad\quad - \xi_j\xi_{j+1}\gamma'(s_j+\eta_j)\wedge O(|s_{j+1}-s_j|+|\eta_{j+1}-\eta_j|) \\
&\qquad\quad + \xi_j^2 O(|s_{j+1}-s_j|+|\eta_{j+1}-\eta_j|)\wedge\gamma''(s_j+\eta_j)\Big)\Big| \\
&\le c\frac{|\eta_{j+1}-\eta_j|}{h_j}\Big(|\xi_j|\,|\xi_{j+1}-\xi_j| + \big(|\xi_j|\,|\xi_{j+1}|+|\xi_j|^2\big) \\
&\qquad\qquad\qquad\qquad\qquad \big(|s_{j+1}-s_j|+|\eta_{j+1}-\eta_j|\big)\Big) \\
&\le ch|\nabla\eta_h|\big(\|\xi_h\|_{C^0}|\nabla\xi_h| + \|\xi_h\|_{C^0}^2(1+|\nabla\eta_h|)\big),
\end{aligned}$$

where pointwise quantities in the last expression are taken on the arc corresponding to the edge E_j.

We similarly estimate

$$\begin{aligned}
&\Big|t\mathcal{E}_1 + \frac{1}{h_j}(\eta_{j+1}-\eta_j)\int_0^1 \gamma'\big(s_j+\eta_j+r(s_{j+1}+\eta_{j+1}-s_j-\eta_j)\big)\,dr \\
&\qquad\qquad\qquad\qquad\qquad \wedge t\xi_{j+1}^2\gamma''(s_{j+1}+\eta_{j+1})\Big|,
\end{aligned}$$

and hence $|\mathcal{E}_1+\mathcal{E}_3|$, by the same quantity, and thence obtain

$$\begin{aligned}
&\big|(w_h\wedge u_{h\tau}+v_h\wedge v_{h\tau}) - (\overline{w}_h\wedge\overline{u}_{h\tau}+\overline{v}_h\wedge\overline{v}_{h\tau})\big| \\
&\qquad \le |\mathcal{E}_1+\mathcal{E}_3| + |\mathcal{E}_2| + |\mathcal{E}_4| \\
&\qquad \le ch|\nabla\eta_h|\big(\|\xi_h\|_{C^0}|\nabla\xi_h| + \|\xi_h\|_{C^0}^2(1+|\nabla\eta_h|)\big) \\
&\qquad\qquad\qquad\qquad + c\|\xi_h\|_{C^0}^2(|\eta_j|+|\eta_{j+1}|).
\end{aligned}$$

It follows that

$$\begin{aligned}
|E_2| &\le ch|\eta_h|_{H^1}\big(|\log h|^{1/2}|\xi_h|_{H^{1/2}}^2 + \|\xi_h\|_{C^0}(1+|\eta_h|_{H^1})\big) \\
&\qquad\qquad\qquad + c\|\xi_h\|_{C^0}\|\eta_h\|_{L^2} \\
&\le c|\eta_h|_{H^{1/2}}\big(|\log h|^{1/2}|\xi_h|_{H^{1/2}}^2 + |\log h|\,|\xi_h|_{H^{1/2}}^2(h^{1/2}+|\eta_h|_{H^{1/2}})\big)
\end{aligned}$$

$$+ c|\log h|\,|\xi_h|^2_{H^{1/2}}|\eta_h|_{H^{1/2}}$$
$$\le c|\log h|\,|\eta_h|_{H^{1/2}}|\xi_h|^2_{H^{1/2}},$$

which together with the estimate for E_1 implies

$$|E| \le c|\log h|^{1/2}|(\eta_h,\chi_h)|_{H^{1/2}\times H^1}|\xi_h|^2_{H^{1/2}}.$$

To estimate F we write

$$\begin{aligned} F &= \int_{\partial D_h} (Q'(u_h)v_h - Q'(\overline{u}_h)\overline{v}_h)\cdot v_h \wedge u_{h\tau} \\ &\quad \int_{\partial D_h} (Q'(\overline{u}_h)\overline{v}_h)\cdot (v_h \wedge u_{h\tau} - \overline{v}_h \wedge \overline{u}_{h\tau} \\ &=: F_1 + F_2 \end{aligned}$$

From (64), Lemmas 4.6 and 4.4, Section 5.1 and (71),

$$\begin{aligned} |F_1| &\le c\big(\|u_h - \overline{u}_h\|_{C^0}\|v_h\|_{C^0} + \|v_h - \overline{v}_h\|_{C^0}\|\overline{u}_h\|_{C^0}\big)\,\|v_h \wedge u_{h\tau}\|_{L^1} \\ &\le c\big(|\log h|\,|(\eta_h,\chi_h)|_{H^{1/2}\times H^1}\,|(\xi_h,\psi_h)|_{H^{1/2}\times H^1} \\ &\qquad + |\log h|\,|\eta_h|_{H^{1/2}}|\xi_h|_{H^{1/2}}\big)h|\xi_h|_{H^{1/2}} \\ &\le ch|\log h|\,|(\xi_h,\psi_h)|^2_{H^{1/2}\times H^1}|(\eta_h,\chi_h)|_{H^{1/2}\times H^1}. \end{aligned}$$

Using (64) and Lemma 4.6 to estimate $\|\overline{v}_h\|_{C^0}$,

$$|F_2| \le c|\log h|\,|(\xi_h,\psi_h)|_{H^{1/2}\times H^1}\|v_h \wedge u_{h\tau} - \overline{v}_h \wedge \overline{u}_{h\tau}\|_{L^1(\partial D_h)}. \tag{74}$$

Using the notation in the estimate for E, we have at $e^{i\theta_j} + t(e^{i\theta_{j+1}} - e^{i\theta_j})$ on the edge E_j,

$$\begin{aligned} &v_h \wedge u_{h\tau} - \overline{v}_h \wedge \overline{u}_{h\tau} \\ &\quad = -\frac{1}{h_j}(\gamma(s_{j+1}) - \gamma(s_j)) \wedge \big((1-t)\xi_j\gamma'(s_j) + t\xi_{j+1}\gamma'(s_{j+1})\big) \\ &\qquad + \frac{1}{h_j}(\gamma(s_{j+1}+\eta_{j+1}) - \gamma(s_j+\eta_j)) \\ &\qquad\qquad \wedge \big((1-t)\xi_j\gamma'(s_j+\eta_j) + t\xi_{j+1}\gamma'(s_{j+1}+\eta_{j+1})\big) \\ &\quad = -\frac{1}{h_j}(s_{j+1}-s_j)\int_0^1 \gamma'(s_j + r(s_{j+1}-s_j))\,dr \\ &\qquad\qquad \wedge \big((1-t)\xi_j\gamma'(s_j) + t\xi_{j+1}\gamma'(s_{j+1})\big) \\ &\qquad + \frac{1}{h_j}(s_{j+1}+\eta_{j+1}-s_j-\eta_j) \\ &\qquad\qquad\qquad \int_0^1 \gamma'(s_j+\eta_j) + r(s_{j+1}+\eta_{j+1}-s_j-\eta_j))\,dr \\ &\qquad\qquad \wedge \big((1-t)\xi_j\gamma'(s_j+\eta_j) + t\xi_{j+1}\gamma'(s_{j+1}+\eta_{j+1})\big) \\ &\quad = \frac{\eta_{j+1}-\eta_j}{h_j}\int_0^1 \gamma'(s_j+\eta_j+r(s_{j+1}+\eta_{j+1}-s_j-\eta_j))\,dr \\ &\qquad\qquad \wedge (1-t)\xi_j\gamma'(s_j+\eta_j) \\ &\qquad + \frac{\eta_{j+1}-\eta_j}{h_j}\int_0^1 \gamma'(s_j+\eta_j+r(s_{j+1}+\eta_{j+1}-s_j-\eta_j))\,dr \\ &\qquad\qquad \wedge (1-t)\xi_{j+1}\gamma'(s_{j+1}+\eta_{j+1}) \end{aligned}$$

$$
\begin{aligned}
&+\frac{s_{j+1}-s_j}{h_j}\Big(\int_0^1 \gamma'(s_j+\eta_j+r(s_{j+1}+\eta_{j+1}-s_j-\eta_j))\,dr\\
&\qquad\qquad\qquad \wedge\big((1-t)\xi_j\gamma'(s_j+\eta_j)+t\xi_{j+1}\gamma'(s_{j+1}+\eta_{j+1})\big)\\
&\qquad\qquad -\int_0^1 \gamma'(s_j+r(s_{j+1}-s_j))\,dr\\
&\qquad\qquad\qquad \wedge\big((1-t)\xi_j\gamma'(s_j)+t\xi_{j+1}\gamma'(s_{j+1})\big)\Big)\\
&=:\overline{F}_1+\overline{F}_2+\overline{F}_3
\end{aligned}
$$

The main point in order to estimate $\overline{F}_1$ and $\overline{F}_2$ is that, after integrating, $(\eta_{j+1}-\eta_j)/h_j$ leads to $|\eta_h|_{H^1}$ ($\le ch^{-1/2}|\eta_h|_{H^{1/2}}$), and so we need to gain a factor $h^{1/2}$. We have

$$
\begin{aligned}
\overline{F}_1 = &-\frac{\eta_{j+1}-\eta_j}{h_j}\,(1-t)\,\xi_j\gamma'(s_j+\eta_j)\\
&\wedge\Big(\gamma'(s_j+\eta_j)+(s_{j+1}+\eta_{j+1}-s_j-\eta_j)\\
&\qquad\int_0^1\int_0^1 r\gamma''(s_j+\eta_j+pr(s_{j+1}+\eta_{j+1}-s_j-\eta_j))\,dr\,dp\Big)
\end{aligned}
$$

and so (evaluating quantities at the point with parameter t)

$$
|\overline{F}_1| \le ch|\eta_h'|(|s_h'|+|\eta_h'|)|\xi_j|.
$$

Similarly,

$$
|\overline{F}_2| \le ch|\eta_h'|(|s_h'|+|\eta_h'|)|\xi_{j+1}|.
$$

Since γ' is Lipschitz, subtracting terms and estimating,

$$
|\overline{F}_3| \le c|s_h'|\,(|\eta_{j+1}|+|\eta_j|)\,(|\xi_j|+|\xi_{j+1}|).
$$

From the pointwise estimates for $|\overline{F}_1|, |\overline{F}_2|, |\overline{F}_3|$, using (40), the inverse estimate from H^1 to $H^{1/2}$, the bound on $|\eta_h|_{H^{1/2}}$ from the hypothesis of the theorem,(64) and (74), it follows that

$$
\begin{aligned}
|F_2| &\le c|\log h|\,|(\xi_h,\psi_h)|_{H^{1/2}\times H^1}\\
&\qquad \Big(h|\eta_h|_{H^1}\big(|s_h|_{H^1}+|\eta_h|_{H^1}\big)\,\|\xi_h\|_{C^0}+|s_h|_{C^1}\|\eta_h\|_{L^2}\|\xi_h\|_{L^2}\Big)\\
&\le c|\log h|\,|(\xi_h,\psi_h)|_{H^{1/2}\times H^1}\\
&\qquad \Big(|\eta_h|_{H^{1/2}}\big(h^{1/2}+|\eta_h|_{H^{1/2}}\big)|\log h|^{1/2}|\xi_h|_{H^{1/2}}+|\eta_h|_{H^{1/2}}|\xi_h|_{H^{1/2}}\Big)\\
&\le c|\log h|^{3/2}|\eta_h|_{H^{1/2}}|(\xi_h,\psi_h)|^2_{H^{1/2}\times H^1}.
\end{aligned}
$$

It now follows that

$$
|F| \le c|\log h|^{3/2}|(\xi_h,\psi_h)|^2_{H^{1/2}\times H^1}|\eta_h|_{H^{1/2}}.
$$

The Proposition follows by combining the estimates for A–F .

The proof of the main theorem is now a consequence of the following quantitative version of the Inverse Function Theorem, as follows for example from the proof in [Be; pp 113–114]. The modifications necessary since $\mathcal{X}$ is an *affine* space are trivial.

Lemma 5.4. *Let $\mathcal{X}$ be an affine Banach space with Banach space X as tangent space, and let Y be a Banach space. Suppose $x_0 \in \mathcal{X}$ and $f \in C^1(\mathcal{X}, Y)$. Assume there are positive constants α, β, δ and ϵ such that*

$$\begin{aligned} \|f(x_0)\|_Y &\le \delta, & (75)\\ \|f'(x_0)^{-1}\|_{L(Y,X)} &\le \alpha^{-1}, & (76)\\ \|f'(x) - f'(x_0)\|_{L(X,Y)} &\le \beta \quad \text{for all } x \in \overline{B}_\epsilon(x_0), & (77) \end{aligned}$$

where

$$\beta < \alpha, \quad \delta \le (\alpha - \beta)\epsilon. \tag{78}$$

Then there exists a unique $x_* \in \overline{B}_\epsilon(x_0)$ such that $f(x_*) = 0$.

In the following, by $\|H\|_{C^2,loc}$ we imply dependence on M as in Section 5.1

Theorem 5.5. *Assume Γ is a simple closed curve in $\mathbb{R}^3$ with C^4 monotone parametrisiation γ . Let $u = \Phi(\gamma \circ s) + \varphi$ be a C^2 nondegenerate conformally parametrised surface spanning Γ, with prescribed mean curvature given by the C^2 function H, with nondegeneracy constant λ and with ν as in* (33).

Then there exist positive constants c_0, h_0 and ϵ_0 depending on $\|\gamma\|_{C^4}$, $\|\,|\gamma'|^{-1}\,\|_{L^\infty}$, $\|s\|_{C^2}$, $\|H\|_{C^2,loc}$, and also on λ, ν in the case of h_0 and ϵ_0, such that if $0 < h \le h_0$ then there is a discrete nondegenerate conformally parametrised surface $u_h = \Phi_h I_h(\gamma \circ s_h) + \varphi_h$ spanning Γ, with discrete mean curvature given by H, satisfying

$$\begin{gathered} \|s - s_h\|_{H^{1/2}(\partial D_h)} \le c_0\lambda^{-1}h, \quad \|\varphi - \varphi_h\|_{H^1(D_h)} \le c_0\lambda^{-1}h, \\ \|u - u_h\|_{H^1(D_h)} \le c_0\lambda^{-1}h. \end{gathered} \tag{79}$$

Moreover, u_h is the unique discrete conformally parametrised surface spanning Γ, with discrete mean curvature given by H, satisfying

$$\|s - s_h\|_{H^{1/2}(\partial D)} \le \epsilon_0|\log h|^{-3/2}, \quad \|\varphi - \varphi_h\|_{H^1(D_h)} \le \epsilon_0|\log h|^{-3/2}.$$

Proof. We apply the previous version of the Inverse Function Theorem with $\mathcal{X} = \mathcal{H}_h \times X_{h0}$, $Y = (H_h \times X_{h0})'$ (the dual space), $f = E_h'$, $f' = E_h''$, and $x_0 = (p_h s, I_h\varphi)$.

From Proposition 5.1, (75) holds with $\delta = c_1 h$.

From Proposition 5.2, setting $h_0 = h_1$ and restricting h to $0 < h \le h_0$, (76) holds with $\alpha = 3\lambda/4$.

From Proposition 5.3, choose ϵ_0 so $|\log h|^{3/2}|(\eta_h, \chi_h)|_{H^{1/2}\times H^1} \le \epsilon_0$ implies $\|E_h''(p_h s, I_h\varphi) - E_h''(p_h s + \eta_h, I_h\varphi + \chi_h)\| \le \lambda/2$. Then (77) holds with $\beta = \lambda/2$ and $\epsilon = \epsilon_0/|\log h|^{3/2}$, i.e. $4c_1 h/\lambda \le \epsilon_0/|\log h|^{3/2}$.

By further restricting h_0 if necessary, (78) also holds and so by Lemma 5.4 there is a unique discrete solution $u_h = \Phi_h I_h(\gamma \circ s_h) + \varphi_h$ as in Definition 3.1 in the range $|(s_h - p_h s, \varphi_h - I_h\varphi)|_{H^{1/2}\times H^1} \le \epsilon_0/|\log h|^{3/2}$, for $0 < h \le h_0$.

Taking $\delta = c_1 h$, $\alpha = 3\lambda/4$, $\beta = \lambda/2$, and $\epsilon = 4c_1 h/\lambda$, the hypotheses of the Inverse Function Theorem again hold for $0 < h \le h_0$. recalling that $4c_1 h/\lambda \le \epsilon_0/|\log h|^{3/2}$, it follows that the unique solution from the previous paragraph also satisfies $|(s_h - p_h s, \varphi_h - I_h\varphi)|_{H^{1/2}\times H^1} \le 4c_1 h/\lambda$.

Using the interpolation estimate (39) completes the proof of the theorem (with possibly new h_0, ϵ_0, c_0), except for the last inequality in (79). This follows from the expressions for u and u_h in terms of (s, φ) and (s_h, φ_h), Proposition 4.1 and the decoupling (23), and is otherwise almost exactly the same as the proof of Theorem 5.5 in [DH4].

6 Numerical Implementation

Functions which are stationary for the discrete energy functional E_h can be found directly by applying a Newton procedure in the class $\mathcal{H}_h \times X_{h0}$. Alternatively we here work in the subclass of pairs (s_h, φ_h) which are already stationary for the E_h with respect to variations fixing s_h, i.e. such that $\partial E_h/\partial\varphi_h(s_h, \varphi_h) = 0$. Defining $u_h = \Phi_h(I_h(\gamma \circ s_h)) + \varphi_h$ as in (3.1), we say u_h is discrete H-harmonic.

For s_h in a neighourhood of a nondegenerate staionary function for E, local uniqueness of u_h and hence of φ_h follows from the non-degeneracy of the discrete functional in Proposition 5.2. Thus if we set

$$F_h(s_h) = E_h(s_h, \varphi_h(s_h))$$

then

$$F_h'(s_h) = \frac{\partial E_h}{\partial s_h}(s_h, \varphi_h(s_h)),$$

and s_h is stationary for F_h iff $(s_h, \varphi_h(s_h))$ is stationary for E_h.

Algorithm 1 (Main Algorithm). For given initial guess $s_h \in \mathcal{H}_h$ and tolerance $\varepsilon > 0$

1. Compute $u_h \in X_h$ with $u_h = I_h(\gamma \circ s_h)$ on ∂D_h and stationary for F_h':
$$\int_{D_h} \nabla u_h \cdot \nabla \psi_h + 2\int_{D_h} H(u_h)\psi_h \cdot u_{hx} \wedge u_{hy} = 0 \quad \forall \varphi_h \in X_{h0}. \tag{80}$$
2. Set up the right hand side of (82):
$$\begin{aligned}\langle F_h'(s_h), \eta_h\rangle = \int_{D_h} &\nabla u_h \cdot \nabla v_h^0 + 2\int_{D_h} H(u_h)v_h^0 \cdot u_{hx} \wedge u_{hy} \\ &-\frac{2}{3}\int_{\partial D_h} Q(u_h) \cdot v_h^0 \wedge u_{h\tau} \quad \forall \eta_h \in H_h,\end{aligned} \tag{81}$$
with v_h^0 any continuation of $I_h(\gamma' \circ s_h\ \eta_h)$ to $\overline{D}_h$.
3. Solve the linear system $\xi_h \in H_h$
$$F_h''(s_h)\xi_h = -F_h'(s_h) \tag{82}$$
with a biconjugate gradient algorithm.
4. If $\|\xi_h\|_H < \varepsilon$, then stop. Otherwise update the solution
$$s_h := s_h + \xi_h \tag{83}$$
and go to step 1.

For the first step we use the Newton method with a suitable initial guess and then with the last u_h available in the algorithm. In order to keep the computational complexity of the problem reasonable, in step 2 we use as a special continuation v_h^0 the discrete function which is zero at all interior nodes in D_h and equals $I_h(\gamma' \circ s_h\eta_h)$ on the boundary ∂D_h. Thus each component of $F_h'(s_h)$ is of complexity nb, where nb is the number of nodes on ∂D_h. The reduction of the general form of $F_h'(s_h)$ to the form which is given in (81) is done with the use of equation (80).

The most important step in the Main Algorithm is the solution of the linear system (82). The standard bicg method requires an algorithm for matrix-vector multiplication. This step is described in the following Algorithm. It shows how the multiplication $F''(s_h)\xi_h$ for an arbitrary $\xi_h \in H_h$ is done. Here again we use the above mentioned reduction of complexity by choosing v_h^0 and w_h^0 from X_h such that they are zero at interior nodes and have the correct boundary data.

Algorithm 2 (Matrix-vector multiplication). For given $u_h \in X_h$, $s_h \in \mathcal{H}_h$ and $\xi_h \in H_h$

1. Compute $v_h \in X_h$ such that $v_h = I_h(\gamma' \circ s_h)$ on ∂D_h and
$$\int_{D_h} \nabla v_h \cdot \nabla \varphi_h + 2\int_{D_h} H(u_h)\,(v_{hx} \wedge u_{hy} + u_{hx} \wedge v_{hy}) \cdot \varphi_h$$
$$+2\int_{D_h} H'(u_h) \cdot v_h\ \varphi_h \cdot u_{hx} \wedge u_{hy} = 0 \quad \forall \varphi_h \in X_{h0}. \tag{84}$$

2. For arbitrary $\eta_h \in H_h$
$$\begin{aligned} F_h''(s_h)(\xi_h, \eta_h) = &\int_{D_h} \nabla u_h \cdot \nabla w_h^0 + \int_{D_h} \nabla v_h \cdot \nabla v_h^0 \\ &+ 2\int_{D_h} H(u_h) w_h^0 \cdot u_{hx} \wedge u_{hy} \\ &+ 2\int_{D_h} H(u_h) v_h^0 \cdot (v_{hx} \wedge u_{hy} + u_{hx} \wedge v_{hy}) \\ &+ 2\int_{D_h} H'(u_h) \cdot v_h v_h^0 \cdot u_{hx} \wedge u_{hy} \\ &- \frac{2}{3}\int_{\partial D_h} Q(u_h) \cdot \left(w_h^0 \wedge u_{h\tau} + v_h^0 \wedge v_{h\tau}\right) \\ &- \frac{2}{3}\int_{\partial D_h} Q'(u_h) v_h \cdot v_h^0 \wedge u_{h\tau}. \end{aligned} \tag{85}$$
where v_h^0 is a continuation of $I_h(\gamma' \circ s_h \eta_h)$ and w_h^0 is a continuation of $I_h(\gamma'' \circ s_h \xi_h \eta_h)$ to $\overline{D}_h$.

The reduction of the general form of the second derivative of F_h to the form used in (85) is done with the use of the equation (84).

We start with a test computation of a problem for which we know exact small and large solutions. See Figure 2 for the discrete solutions. That the large solution appears relatively

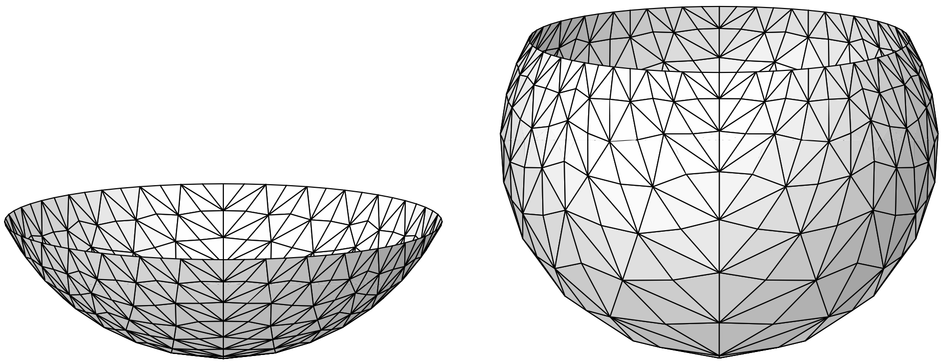

Figure 2: Small and large solution for circular boundary with $H = 0.9$ with 289 nodes.

coarsely approximated is a consequence of the conformal parametrisation of the surface. Let the constant prescribed mean curvature H be positive and less than 1. The boundary curve is $\Gamma = S^1$, the parametrisation s is given by $s(\varphi) = \varphi$ and
$$u_j(x) = \frac{1+\delta^2}{|x|^2+\delta^2} x_j \ (j = 1, 2), \quad u_3(x) = \delta \frac{|x|^2 - 1}{|x|^2+\delta^2}, \tag{86}$$

h	nodes	$L^\infty(D)$	*eoc*	$L^2(D)$	*eoc*	$H^1(D)$	*eoc*
1.0	9	0.1657e-1	-	0.1002	-	0.2607	-
0.7368	25	0.1616e-1	0.08	0.3900e-1	3.09	0.1816	1.18
0.4203	81	0.4568e-2	2.25	0.1070e-1	2.30	0.9624e-1	1.13
0.2219	289	0.1231e-2	2.05	0.2703e-2	2.15	0.4816e-1	1.08
0.1137	1089	0.3349e-3	1.95	0.6724e-3	2.08	0.2389e-1	1.05
0.5754e-1	4225	0.8949e-4	1.94	0.1673e-3	2.04	0.1187e-1	1.03

Table 1: Absolute errors for the small solution from (86) with $H = 0.5$.

h	nodes	$L^\infty(D)$	*eoc*	$L^2(D)$	*eoc*	$H^1(D)$	*eoc*
0.4203	81	0.1299e1	-	0.1230e1	-	0.6188e1	-
0.2219	289	0.5629	1.31	0.4679	1.51	0.2908e1	1.18
0.1137	1089	0.1904	1.62	0.1611	1.60	0.1137e1	1.40
0.5754e-1	4225	0.5785e-1	1.75	0.4708e-1	1.81	0.5870	0.97

Table 2: Absolute errors for the Absolute errors for the large solution from (86) with $H = 0.5$.

where $\delta = H/(H^2 + \sqrt{1-H^2})$ for the small and $\delta = H/(H^2 - \sqrt{1-H^2})$ for the large solution respectively.

In Tables 1 and 2 we list the errors between the smooth solution u and the discrete solution u_h in the norms $L^\infty(D)$, $L^2(D)$ and $H^1(D)$ for different grid sizes. From the errors $e(h_1)$ and $e(h_2)$ for two successive grid sizes h_1 and h_2 for a certain norm we computed the experimental order of convergence

$$eoc = \log \frac{e(h_1)}{e(h_2)} \Big/ \log \frac{h_1}{h_2}.$$

The results show that the theoretically proved order of convergence in the $H^1(D)$-norm is reproduced by the practical computations. In Figure 1 we have shown two solutions of the Plateau Problem for the same boundary curve

$$\gamma(s) = ((1 + 0.1\cos 3s)\cos 2s, (1 + 0.1\cos 3s)\sin 2s, -\sin s)$$

but with the different constant curvatures $H = 0$ and $H = 0.8$. Figure 3 shows the depen-

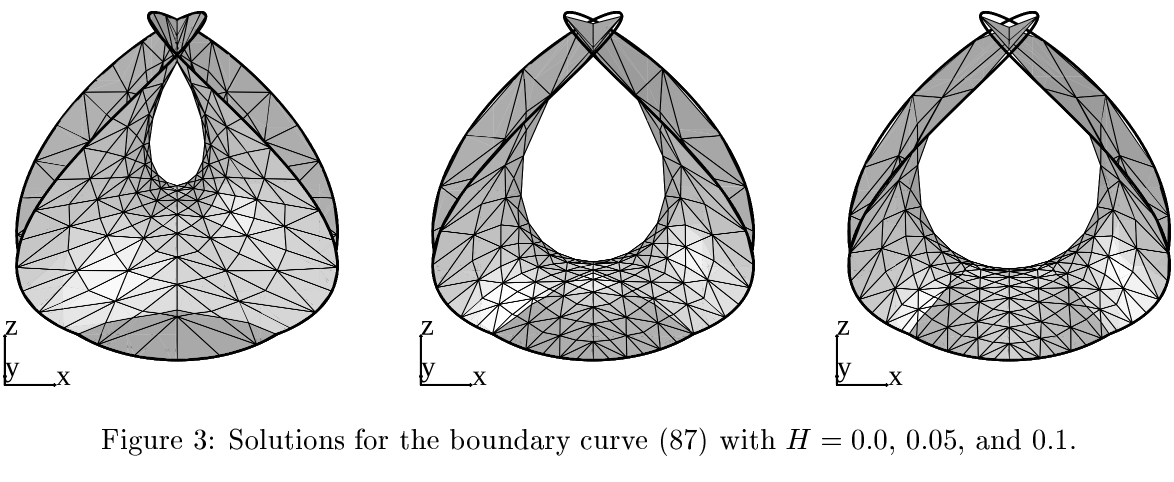

Figure 3: Solutions for the boundary curve (87) with $H = 0.0$, 0.05, and 0.1.

dency of solutions on mean curvature for the boundary curve

$$\gamma(s) = \left(R\cos s - \frac{R^3}{3}\cos 3s, R\sin s + \frac{R^3}{3}\sin 3s, R^2\cos 2s \right). \tag{87}$$

with $R = 2$.

In Figure 4 we show a small and a large solution spanned in the curve (87) for $R = 0.75$ with constant mean curvature $H = 0.5$. The corresponding parametrisations $s = s(\varphi)$ are plotted in Figure 5 and exhibit interesting symmetries.

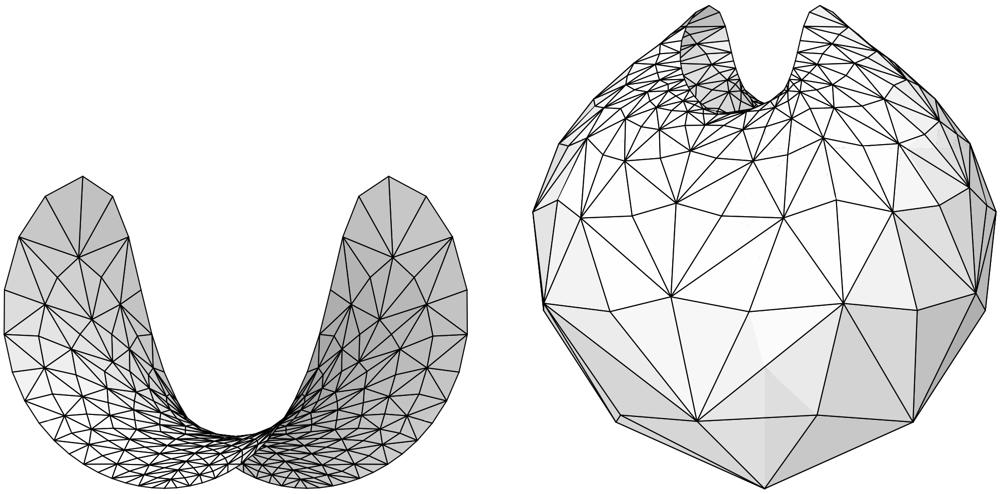

Figure 4: Small (graphically enlarged) and large solutions for the boundary curve (87) with $H = 0.5$

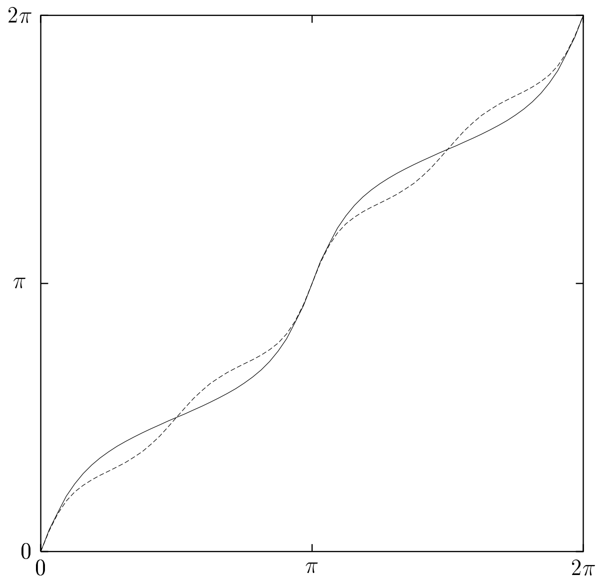

Figure 5: Parametrisations for the small and large (dashed line) solution from Figure 4.

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