Homogenization in the Hencky plasticity setting

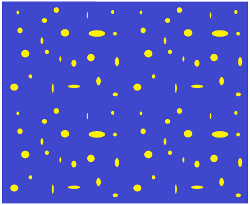
Martin Jesenko

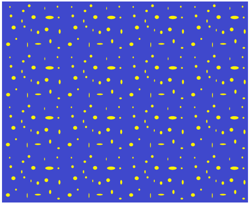
joint work with Bernd Schmidt (Universität Augsburg)

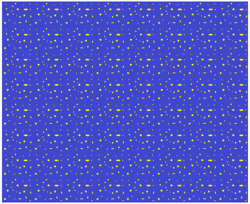
M. Jesenko, B. Schmidt, *Homogenization and the limit of vanishing hardening in Hencky plasticity with non-convex potentials*, arXiv:1703.09443 [math.AP]

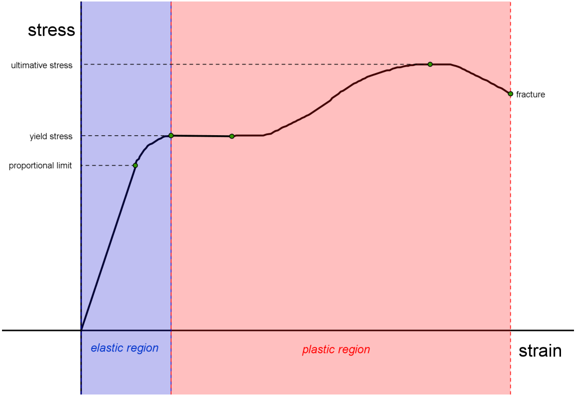
Homogenization Theory and Applications (HomTAp)
Berlin, October 4-6, 2017

Homogenization and stress-strain diagram

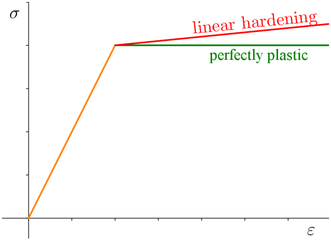








Hencky plasticity

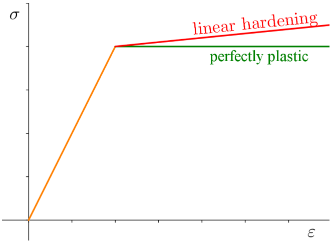


Our model:

- elastic regime with linear dependence,
- perfectly plastic regime (Hencky plasticity),
- plastic regime with linear hardening.

Elastic region K is determined by some yield criterion (von Mises, Tresca, . . .).

Hencky plasticity



Our model:

- elastic regime with linear dependence,
- perfectly plastic regime (Hencky plasticity),
- plastic regime with linear hardening.

Elastic region K is determined by some yield criterion (von Mises, Tresca, ...).

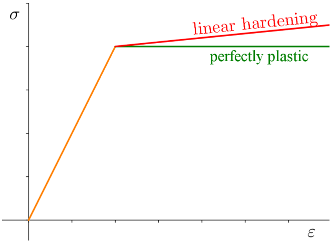
Thus, the energy at zero hardening is given by

$$\mathcal{F}(u) = \int_\Omega f(\mathfrak{E}u(x))\, dx \quad \text{with} \quad f(X) = f_{\text{dev}}(X_{\text{dev}}) + \frac{\varkappa}{2}(\operatorname{tr} X)^2$$

where $\mathfrak{E}u(x) = \frac{1}{2}(\nabla u(x) + \nabla u(x)^T)$, $X_{\text{dev}} = X - \frac{\operatorname{tr} X}{n} I$ and f_{dev} is convex and given by

$$f^*_{\text{dev}}(\sigma_{\text{dev}}) = \begin{cases} \frac{1}{4\mu}|\sigma_{\text{dev}}|^2, & \sigma_{\text{dev}} \in K_{\text{dev}}, \\ \infty, & \sigma_{\text{dev}} \notin K_{\text{dev}}. \end{cases}$$

Hencky plasticity



Our model:

- elastic regime with linear dependence,
- perfectly plastic regime (Hencky plasticity),
- plastic regime with linear hardening.

Elastic region K is determined by some yield criterion (von Mises, Tresca, ...).

Thus, the energy at zero hardening is given by

$$\mathcal{F}(u) = \int_\Omega f(\mathfrak{E}u(x))\, dx \quad \text{with} \quad f(X) = f_{\text{dev}}(X_{\text{dev}}) + \frac{\varkappa}{2}(\operatorname{tr} X)^2$$

where $\mathfrak{E}u(x) = \frac{1}{2}(\nabla u(x) + \nabla u(x)^T)$, $X_{\text{dev}} = X - \frac{\operatorname{tr} X}{n} I$ and f_{dev} is convex and given by

$$f^*_{\text{dev}}(\sigma_{\text{dev}}) = \begin{cases} \frac{1}{4\mu}|\sigma_{\text{dev}}|^2, & \sigma_{\text{dev}} \in K_{\text{dev}}, \\ \infty, & \sigma_{\text{dev}} \notin K_{\text{dev}}. \end{cases}$$

f_{dev} has linear growth. Thus f grows linearly in the deviatoric direction and quadratically in the trace.

Solvability shown e.g. in [Anzellotti, Giaquinta 82]

Non-homogeneous convex setting considered in [Demengel, Qi 90]

Plan of the talk

- Setting and spaces
- Case with hardening
- Homogenized density
- Recovery sequence
- liminf-inequality

Setting

Generalization: non-convex non-homogenous energy

Let $f : \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\text{sym}} \to \mathbb{R}$

- be $\mathbb{I}^n$-periodic Carathéodory function and
- have a Hencky plasticity growth, i.e. $\exists \alpha, \beta > 0$ such that for all $x \in \Omega$ and $X \in \mathbb{R}^{n\times n}_{\text{sym}}$

$$\alpha\big(|X_{\text{dev}}| + (\operatorname{tr} X)^2\big) \leq f(x, X) \leq \beta\big(|X_{\text{dev}}| + (\operatorname{tr} X)^2 + 1\big).$$

Let $\mathcal{F}_\varepsilon : L^1(\Omega; \mathbb{R}^n) \to \mathbb{R} \cup \{\infty\}$ be defined by

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_\Omega f\big(\frac{x}{\varepsilon}, \mathfrak{E}u(x)\big)\, dx, & u \in ???, \\ \infty, & \text{else.} \end{cases}$$

Does $\{\mathcal{F}_\varepsilon\}_\varepsilon$ Γ-converge and whereto?

Setting

Generalization: non-convex non-homogenous energy

Let $f : \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$

- be $\mathbb{I}^n$-periodic Carathéodory function and
- have a Hencky plasticity growth, i.e. $\exists \alpha, \beta > 0$ such that for all $x \in \Omega$ and $X \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$

$$\alpha\big(|X_{\mathrm{dev}}| + (\operatorname{tr} X)^2\big) \leq f(x, X) \leq \beta\big(|X_{\mathrm{dev}}| + (\operatorname{tr} X)^2 + 1\big).$$

Let $\mathcal{F}_\varepsilon : L^1(\Omega; \mathbb{R}^n) \to \mathbb{R} \cup \{\infty\}$ be defined by

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_\Omega f\big(\frac{x}{\varepsilon}, \mathfrak{E}u(x)\big)\, dx, & u \in ???, \\ \infty, & \text{else.} \end{cases}$$

Does $\{\mathcal{F}_\varepsilon\}_\varepsilon$ Γ-converge and whereto?

Conjecture: The limit is of form

$$\mathcal{F}(u) := \begin{cases} \int_\Omega f_{\mathrm{hom}}\big(\mathfrak{E}u(x)\big)\, dx + \text{recession(singular part)}, & u \in \text{ wk-cl(???)}, \\ \infty, & \text{else.} \end{cases}$$

1. Symmetrized gradient must exist (in weak sense), and

$$LD(\Omega;\mathbb{R}^n) := \{u \in L^1(\Omega;\mathbb{R}^n) : \mathfrak{E}u \in L^1(\Omega;\mathbb{R}^{n\times n})\}, \quad \|u\|_{LD} := \|u\|_{L^1} + \|\mathfrak{E}u\|_{L^1}.$$

Setting
Domain

1. Symmetrized gradient must exist (in weak sense), and

$$LD(\Omega;\mathbb{R}^n) := \{u \in L^1(\Omega;\mathbb{R}^n) : \mathfrak{E}u \in L^1(\Omega;\mathbb{R}^{n\times n})\}, \quad \|u\|_{LD} := \|u\|_{L^1} + \|\mathfrak{E}u\|_{L^1}.$$

2. The natural domain for $\mathcal{F}$ is

$$LU(\Omega;\mathbb{R}^n) := \{u \in LD(\Omega;\mathbb{R}^n) : \operatorname{div} u \in L^2(\Omega)\}, \quad \|u\|_{LU} := \|u\|_{LD} + \|\operatorname{div} u\|_{L^2}.$$

Setting
Domain

1. Symmetrized gradient must exist (in weak sense), and

$$LD(\Omega;\mathbb{R}^n) := \{u \in L^1(\Omega;\mathbb{R}^n) : \mathfrak{E}u \in L^1(\Omega;\mathbb{R}^{n\times n})\}, \quad \|u\|_{LD} := \|u\|_{L^1} + \|\mathfrak{E}u\|_{L^1}.$$

2. The natural domain for $\mathcal{F}$ is

$$LU(\Omega;\mathbb{R}^n) := \{u \in LD(\Omega;\mathbb{R}^n) : \operatorname{div} u \in L^2(\Omega)\}, \quad \|u\|_{LU} := \|u\|_{LD} + \|\operatorname{div} u\|_{L^2}.$$

3. Due to the lack of weak compactness, we introduce the space $BD(\Omega;\mathbb{R}^n)$ of all $u \in L^1(\Omega;\mathbb{R}^n)$ such that $Eu \in M(\Omega;\mathbb{R}^{n\times n})$ with the norm

$$\|u\|_{BD} = \|u\|_{L^1} + \|Eu\|_M.$$

Then

$$Eu = \mathfrak{E}u\,\mathcal{L}^n + E^s u.$$

Setting
Domain

1. Symmetrized gradient must exist (in weak sense), and

$$LD(\Omega;\mathbb{R}^n) := \{u \in L^1(\Omega;\mathbb{R}^n) : \mathfrak{E}u \in L^1(\Omega;\mathbb{R}^{n\times n})\}, \quad \|u\|_{LD} := \|u\|_{L^1} + \|\mathfrak{E}u\|_{L^1}.$$

2. The natural domain for $\mathcal{F}$ is

$$LU(\Omega;\mathbb{R}^n) := \{u \in LD(\Omega;\mathbb{R}^n) : \operatorname{div} u \in L^2(\Omega)\}, \quad \|u\|_{LU} := \|u\|_{LD} + \|\operatorname{div} u\|_{L^2}.$$

3. Due to the lack of weak compactness, we introduce the space $BD(\Omega;\mathbb{R}^n)$ of all $u \in L^1(\Omega;\mathbb{R}^n)$ such that $Eu \in M(\Omega;\mathbb{R}^{n\times n})$ with the norm

$$\|u\|_{BD} = \|u\|_{L^1} + \|Eu\|_M.$$

Then

$$Eu = \mathfrak{E}u\,\mathcal{L}^n + E^s u.$$

4. Moreover,

$$U(\Omega;\mathbb{R}^n) := \{u \in BD(\Omega;\mathbb{R}^n) : \operatorname{div} u \in L^2(\Omega)\}, \quad \|u\|_U := \|u\|_{BD} + \|\operatorname{div} u\|_{L^2}.$$

Setting
Domain

1. Symmetrized gradient must exist (in weak sense), and

$$LD(\Omega;\mathbb{R}^n) := \{u \in L^1(\Omega;\mathbb{R}^n) : \mathfrak{E}u \in L^1(\Omega;\mathbb{R}^{n\times n})\}, \quad \|u\|_{LD} := \|u\|_{L^1} + \|\mathfrak{E}u\|_{L^1}.$$

2. The natural domain for $\mathcal{F}$ is

$$LU(\Omega;\mathbb{R}^n) := \{u \in LD(\Omega;\mathbb{R}^n) : \operatorname{div} u \in L^2(\Omega)\}, \quad \|u\|_{LU} := \|u\|_{LD} + \|\operatorname{div} u\|_{L^2}.$$

3. Due to the lack of weak compactness, we introduce the space $BD(\Omega;\mathbb{R}^n)$ of all $u \in L^1(\Omega;\mathbb{R}^n)$ such that $Eu \in M(\Omega;\mathbb{R}^{n\times n})$ with the norm

$$\|u\|_{BD} = \|u\|_{L^1} + \|Eu\|_M.$$

   Then

$$Eu = \mathfrak{E}u\ \mathcal{L}^n + E^s u.$$

4. Moreover,

$$U(\Omega;\mathbb{R}^n) := \{u \in BD(\Omega;\mathbb{R}^n) : \operatorname{div} u \in L^2(\Omega)\}, \quad \|u\|_U := \|u\|_{BD} + \|\operatorname{div} u\|_{L^2}.$$

   $c \geq 0$ convex function with linear upper bound. *c-strict convergence*:

   - *strict convergence*: $u_j \to u$ in $L^1(\Omega;\mathbb{R}^n)$ and $|Eu_j|(\Omega) \to |Eu|(\Omega)$,
   - $\operatorname{div} u_j \to \operatorname{div} u$ in $L^2(\Omega)$,
   - $\int_\Omega c(E_{\text{dev}} u_j) \to \int_\Omega c(E_{\text{dev}} u)$ and $\int_\Omega c(Eu_j) \to \int_\Omega c(Eu)$.

Case with hardening

Consider also $f^{(\delta)}(x,X) = f(x,X) + \delta|X_{\mathrm{dev}}|^2$ and

$$\mathcal{F}_\varepsilon^{(\delta)}(u) := \begin{cases} \int_\Omega f^{(\delta)}(\frac{x}{\varepsilon}, \mathfrak{E}u(x))\, dx, & u \in W^{1,2}(\Omega;\mathbb{R}^n), \\ \infty, & \text{else.} \end{cases}$$

For $\delta > 0$ the densities have a quadratic growth in $|X_{\mathrm{sym}}|$. The functionals are therefore of Gårding type ([Schmidt, MJ 14]).

Case with hardening

Consider also $f^{(\delta)}(x,X) = f(x,X) + \delta|X_{\text{dev}}|^2$ and

$$\mathcal{F}_\varepsilon^{(\delta)}(u) := \begin{cases} \int_\Omega f^{(\delta)}(\frac{x}{\varepsilon}, \mathfrak{E}u(x))\,dx, & u \in W^{1,2}(\Omega;\mathbb{R}^n), \\ \infty, & \text{else.} \end{cases}$$

For $\delta > 0$ the densities have a quadratic growth in $|X_{\text{sym}}|$. The functionals are therefore of Gårding type ([Schmidt, MJ 14]). Hence,

$$\Gamma(L^2)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{(\delta)} = \mathcal{F}_{\text{hom}}^{(\delta)}$$

where $\mathcal{F}_{\text{hom}}^{(\delta)}$ has domain $W^{1,2}(\Omega;\mathbb{R}^n)$ and has the density

$$f_{\text{hom}}^{(\delta)}(X) = \inf_{k\in\mathbb{N}} \inf_{\varphi \in W_0^{1,2}(k\mathbb{I}^n;\mathbb{R}^n)} \frac{1}{k^n} \int_{k\mathbb{I}^n} f^{(\delta)}(x, X + \mathfrak{E}\varphi(x))\,dx.$$

In fact, it is also Γ-convergence in L^1 because of the quadratic growth of the density and Poincaré's and Korn's inequality.

Diagrams

$$
\begin{array}{ccc}
\mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\text{pt. falling}} & \mathcal{F}_\varepsilon^{(0)} \\
\Big\downarrow{\scriptstyle \Gamma(L^1)} & & \\
\mathcal{F}_{\text{hom}}^{(\delta)} & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\Gamma(L^1)} & \operatorname{lsc} \mathcal{F}_\varepsilon^{(0)} = \operatorname{lsc} \mathcal{F}_\varepsilon \\
\Big\downarrow{\scriptstyle \Gamma(L^1)} & & \\
\mathcal{F}_{\text{hom}}^{(\delta)} & &
\end{array}
$$

Diagrams

$$\begin{array}{ccc} \mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\text{pt. falling}} & \mathcal{F}_\varepsilon^{(0)} \\ \Big\downarrow {\scriptstyle \Gamma(L^1)} & & \\ \mathcal{F}_{\mathrm{hom}}^{(\delta)} & & \end{array} \qquad \begin{array}{ccc} \mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\Gamma(L^1)} & \operatorname{lsc}\mathcal{F}_\varepsilon^{(0)} = \operatorname{lsc}\mathcal{F}_\varepsilon \\ \Big\downarrow {\scriptstyle \Gamma(L^1)} & & \\ \mathcal{F}_{\mathrm{hom}}^{(\delta)} & & \end{array}$$

Let

$$f_{\mathrm{hom}}(X) := \inf_{k\in\mathbb{N}} \inf_{\varphi\in C_c^\infty(k\mathbb{I}^n;\mathbb{R}^n)} \frac{1}{k^n} \int_{k\mathbb{I}^n} f\big(x, X + \mathfrak{E}\varphi(x)\big)\, dx.$$

Clearly

$$f_{\mathrm{hom}}(X) = \inf_{\delta>0} f_{\mathrm{hom}}^{(\delta)}(X).$$

Define

$$\mathcal{G}^{(0)}(u) := \begin{cases} \int_\Omega f_{\mathrm{hom}}\big(\mathfrak{E}u(x)\big), & u \in LU(\Omega;\mathbb{R}^n) \cap W^{1,2}(\Omega;\mathbb{R}^n), \\ \infty, & \text{else,} \end{cases}$$

Diagrams

$$\begin{array}{ccc} \mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\text{pt. falling}} & \mathcal{F}_\varepsilon^{(0)} \\ \Big\downarrow \Gamma(L^1) & & \\ \mathcal{F}_{\text{hom}}^{(\delta)} & \xrightarrow[\text{pt. falling}]{} & \mathcal{G}^{(0)} \end{array} \qquad \begin{array}{ccc} \mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\Gamma(L^1)} & \operatorname{lsc}\mathcal{F}_\varepsilon^{(0)} = \operatorname{lsc}\mathcal{F}_\varepsilon \\ \Big\downarrow \Gamma(L^1) & & \\ \mathcal{F}_{\text{hom}}^{(\delta)} & \xrightarrow[\Gamma(L^1)]{} & \operatorname{lsc}\mathcal{G}^{(0)} = \operatorname{lsc}\mathcal{G} \end{array}$$

Let

$$f_{\text{hom}}(X) := \inf_{k\in\mathbb{N}} \inf_{\varphi\in C_c^\infty(k\mathbb{I}^n;\mathbb{R}^n)} \frac{1}{k^n}\int_{k\mathbb{I}^n} f\big(x, X + \mathfrak{E}\varphi(x)\big)\,dx.$$

Clearly

$$f_{\text{hom}}(X) = \inf_{\delta>0} f_{\text{hom}}^{(\delta)}(X).$$

Define

$$\mathcal{G}^{(0)}(u) := \begin{cases} \int_\Omega f_{\text{hom}}\big(\mathfrak{E}u(x)\big), & u \in LU(\Omega;\mathbb{R}^n)\cap W^{1,2}(\Omega;\mathbb{R}^n), \\ \infty, & \text{else}, \end{cases}$$

Diagrams

$$\begin{array}{ccc} \mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\text{pt. falling}} & \mathcal{F}_\varepsilon^{(0)} \\ \Big\downarrow {\scriptstyle \Gamma(L^1)} & & \\ \mathcal{F}_{\mathrm{hom}}^{(\delta)} & \xrightarrow[\text{pt. falling}]{} & \mathcal{G}^{(0)} \end{array} \qquad \begin{array}{ccc} \mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\Gamma(L^1)} & \operatorname{lsc} \mathcal{F}_\varepsilon^{(0)} = \operatorname{lsc} \mathcal{F}_\varepsilon \\ \Big\downarrow {\scriptstyle \Gamma(L^1)} & & \\ \mathcal{F}_{\mathrm{hom}}^{(\delta)} & \xrightarrow[\Gamma(L^1)]{} & \operatorname{lsc} \mathcal{G}^{(0)} = \operatorname{lsc} \mathcal{G} \end{array}$$

Let

$$f_{\mathrm{hom}}(X) := \inf_{k \in \mathbb{N}} \inf_{\varphi \in C_c^\infty(k\mathbb{I}^n;\mathbb{R}^n)} \frac{1}{k^n} \int_{k\mathbb{I}^n} f\big(x, X + \mathfrak{E}\varphi(x)\big)\, dx.$$

Clearly

$$f_{\mathrm{hom}}(X) = \inf_{\delta > 0} f_{\mathrm{hom}}^{(\delta)}(X).$$

Define

$$\mathcal{G}^{(0)}(u) := \begin{cases} \int_\Omega f_{\mathrm{hom}}\big(\mathfrak{E}u(x)\big), & u \in LU(\Omega;\mathbb{R}^n) \cap W^{1,2}(\Omega;\mathbb{R}^n), \\ \infty, & \text{else}, \end{cases}$$

Notice

$$\mathcal{F}_{\mathrm{hom}}^{(\delta)} \geq \Gamma\text{-}\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon \quad \text{and therefore} \quad \operatorname{lsc} \mathcal{G} \geq \Gamma\text{-}\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon.$$

Homogenized density

The homogenized density

$$f_{\mathrm{hom}}(X) := \inf_{k\in\mathbb{N}} \inf_{\varphi\in C_c^\infty(k\mathbb{I}^n;\mathbb{R}^n)} \frac{1}{k^n}\int_{k\mathbb{I}^n} f\big(x, X+\mathfrak{E}\varphi(x)\big)\,dx$$

- is symmetric-quasiconvex,
- has Hencky plasticity growth,
- $(f^{\mathrm{qcls}})_{\mathrm{hom}} = f_{\mathrm{hom}}$.

Subadditive $\mathbb{Z}^n$-invariant processes [Akcoglu, Krengel 80], [Licht, Michaille 02]:

$$\inf_{k\in\mathbb{N}} \ldots = \lim_{k\to\infty} \ldots$$

For every open bounded convex set V and $\varepsilon_k \searrow 0$

$$\begin{aligned} f_{\mathrm{hom}}(X) &= \lim_{k\to\infty} \inf_{\varphi\in C_c^\infty(\varepsilon_k^{-1}V;\mathbb{R}^n)} \frac{1}{|\varepsilon_k^{-1}V|}\int_{\varepsilon_k^{-1}V} f(x, X+\mathfrak{E}\varphi(x))\,dx \\ &= \lim_{k\to\infty} \inf_{\varphi\in LU_0(\varepsilon_k^{-1}V;\mathbb{R}^n)} \frac{1}{|\varepsilon_k^{-1}V|}\int_{\varepsilon_k^{-1}V} f(x, X+\mathfrak{E}\varphi(x))\,dx. \end{aligned}$$

Reshetnyak continuity theorem (Kristensen, Rindler 10)

Let $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$, *and*

$$\mu_j \overset{*}{\rightharpoonup} \mu \quad \text{in } M(\Omega; \mathbb{R}^N) \quad \text{and} \quad \langle \mu_j \rangle(\Omega) \to \langle \mu \rangle(\Omega).$$

Then

$$\lim_{j\to\infty} \left[\int_\Omega f\left(x, \frac{d\mu_j^a}{d\mathcal{L}^n}(x)\right) \, dx + \int_\Omega f^\infty \left(x, \frac{d\mu_j^s}{d|\mu_j^s|}(x)\right) \, d|\mu_j^s|(x) \right] =$$

$$= \int_\Omega f\left(x, \frac{d\mu^a}{d\mathcal{L}^n}(x)\right) \, dx + \int_\Omega f^\infty \left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) \, d|\mu^s|(x).$$

$\langle A \rangle := \sqrt{1 + |A|^2}$

$\mathbf{E}(\Omega; \mathbb{R}^N) = \{\text{functions extendable to } \infty\}$

$$g^\infty(X) = \limsup_{Y \to X,\ t \to \infty} \frac{g(tY)}{t}$$

Theorem

$LU(\Omega; \mathbb{R}^n) \cap C^\infty(\Omega; \mathbb{R}^n)$ *is dense in* $U(\Omega; \mathbb{R}^n)$ *in* $\langle \cdot \rangle$*-strict topology.*

Theorem

Let $f : \Omega \times \mathbb{R}^{n\times n}_{\text{sym}} \to \mathbb{R}$ be a continuous function that

- *is symmetric-rank-one-convex in the second variable,*
- *satisfies the Hencky growth condition.*

Denote $f_{\text{dev}} := f|_{\Omega\times\mathbb{R}^{n\times n}_{\text{dev}}}$. Suppose that

$$(f_{\text{dev}})^\infty(x_0, P_0) = \limsup_{P\to P_0, t\to\infty} \frac{f_{\text{dev}}(x_0, tP)}{t}$$

is for every fixed $P_0 \in \mathbb{R}^{n\times n}_{\text{dev}}$ a continuous function of x_0. Then the functional

$$\mathcal{F}(u) = \int_\Omega f\big(x, \mathfrak{E}u(x)\big)\, dx + \int_\Omega (f_{\text{dev}})^\infty\big(x, \tfrac{dE^s u}{d|E^s u|}(x)\big)\, d|E^s u|(x)$$

is $\langle\cdot\rangle$-strictly continuous on $U(\Omega; \mathbb{R}^n)$.

Theorem

Let $f : \Omega \times \mathbb{R}^{n\times n}_{\text{sym}} \to \mathbb{R}$ be a continuous function that

- *is symmetric-rank-one-convex in the second variable,*
- *satisfies the Hencky growth condition.*

Denote $f_{\text{dev}} := f|_{\Omega\times\mathbb{R}^{n\times n}_{\text{dev}}}$. Suppose that

$$(f_{\text{dev}})^\infty(x_0, P_0) = \limsup_{P\to P_0, t\to\infty} \frac{f_{\text{dev}}(x_0, tP)}{t}$$

is for every fixed $P_0 \in \mathbb{R}^{n\times n}_{\text{dev}}$ a continuous function of x_0. Then the functional

$$\mathcal{F}(u) = \int_\Omega f\big(x, \mathfrak{E}u(x)\big)\, dx + \int_\Omega (f_{\text{dev}})^\infty\big(x, \tfrac{dE^s u}{d|E^s u|}(x)\big)\, d|E^s u|(x)$$

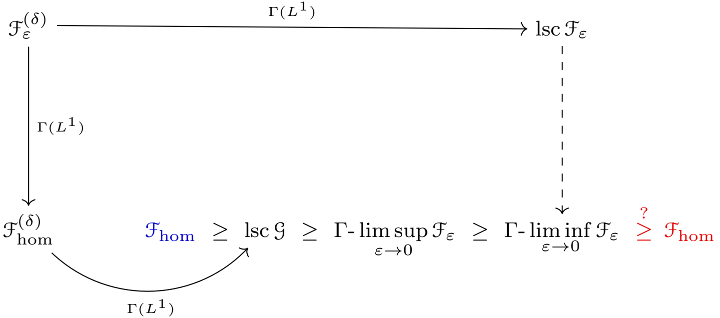
is $\langle\cdot\rangle$-strictly continuous on $U(\Omega;\mathbb{R}^n)$.

Ingredients of the proof:

- Special Lipschitz continuity in the trace direction
- Approximation of functions $\geq -\alpha(1+|X|)$ by functions from $\mathbf{E}(\Omega;\mathbb{R}^N)$ [Alibert, Bouchitté 97]
- Rank-one theorem (De Philippis, Rindler 16): *Let $u \in BD(\Omega;\mathbb{R}^n)$. Then, for $|E^s u|$-a.e. $x \in \Omega$, there exist $a(x), b(x) \in \mathbb{R}^n \setminus \{0\}$ such that*

$$\frac{dE^s u}{d|E^s u|} = a(x) \odot b(x) = \tfrac{1}{2}(a(x)\otimes b(x) + b(x)\otimes a(x)).$$

We now have



with

$$\mathcal{F}_{\mathrm{hom}}(u) := \begin{cases} \int_\Omega f_{\mathrm{hom}}\big(\mathfrak{E}u(x)\big)\,dx + \int_\Omega (f_{\mathrm{hom}})^{\#}\big(\frac{dE^s u}{d|E^s u|}(x)\big)\,d|E^s u|(x), & u \in U(\Omega;\mathbb{R}^n), \\ \infty, & \text{else.} \end{cases}$$

and

$$g^{\#}(X) := \limsup_{t\to\infty} \frac{g(tX)}{t}.$$

lim-inf inequality

We may suppose $\liminf_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j) < \infty$. Let us fix some $1 < q < \frac{n}{n-1}$ and define measures

$$\mu_j := f(\tfrac{\cdot}{\varepsilon_j}, \mathfrak{E}u_j(\cdot))\mathcal{L}^n.$$

lim-inf inequality

We may suppose $\liminf_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j) < \infty$. Let us fix some $1 < q < \frac{n}{n-1}$ and define measures

$$\mu_j := f(\tfrac{\cdot}{\varepsilon_j}, \mathfrak{E}u_j(\cdot))\mathcal{L}^n.$$

By stepwise extracting appropriate subsequences we may get a (not relabeled) sequence such that

- $\lim_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j)$ equals the lim inf above with all $u_j \in LU(\Omega;\mathbb{R}^n)$,
- $u_j \to u$ in $L^q(\Omega;\mathbb{R}^n)$ due to the lower bound on f and since LU is compactly embedded in L^q,
- and $\mu_j \overset{*}{\rightharpoonup} \mu$ in $M(\Omega;\mathbb{R}^n)$.

lim-inf inequality

We may suppose $\liminf_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j) < \infty$. Let us fix some $1 < q < \frac{n}{n-1}$ and define measures

$$\mu_j := f(\tfrac{\cdot}{\varepsilon_j}, \mathfrak{E}u_j(\cdot))\mathcal{L}^n.$$

By stepwise extracting appropriate subsequences we may get a (not relabeled) sequence such that

- $\lim_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j)$ equals the lim inf above with all $u_j \in LU(\Omega; \mathbb{R}^n)$,
- $u_j \to u$ in $L^q(\Omega; \mathbb{R}^n)$ due to the lower bound on f and since LU is compactly embedded in L^q,
- and $\mu_j \overset{*}{\rightharpoonup} \mu$ in $M(\Omega; \mathbb{R}^n)$.

Let

$$\mu = g\mathcal{L}^n + \mu^s.$$

Goal:

- Regular points: for a.e. $x_0 \in \Omega$

$$g(x_0) = \lim_{\rho\to 0} \lim_{j\to\infty} \frac{\mu_j(B_\rho(x_0))}{|B_\rho(x_0)|} \geq f_{\text{hom}}(\mathfrak{E}u(x_0)).$$

- Singular points:

$$\mu^s \geq (f_{\text{hom}})^{\#}\left(\tfrac{dE^s u}{d|E^s u|}\right)|E^s u|.$$

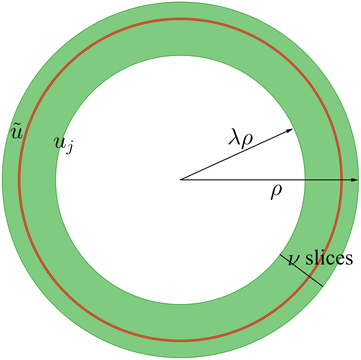
Theorem

Every $u \in BD(\Omega;\mathbb{R}^n)$ is L^q-differentiable a.e. for any $1 \le q \le \frac{n}{n-1}$, i.e., there exists a negligible set $N \subset \Omega$ such that for all $x_0 \in \Omega \setminus N$ there exists a matrix $L_{x_0} \in \mathbb{R}^{n\times n}$ such that

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B_r(x_0)} \left| \frac{u(x) - u(x_0) - L_{x_0}(x - x_0)}{r} \right|^{\frac{n}{n-1}} dx = 0.$$

Therefore, u is a.e. approximately differentiable with $L_{x_0} = \nabla u(x_0)$ being the approximate differential.

Proof: $q = 1$ by [Ambrosio, Coscia, Dal Maso 97] + (Korn-)Poincaré inequality for $BD(\Omega;\mathbb{R}^n)$

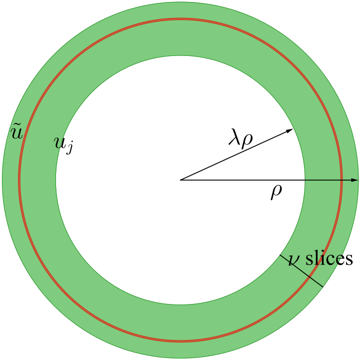


Let us take and fix any x_0 where the function u is approximately differentiable and define

$$\tilde{u}(x) := u(x_0) + \nabla u(x_0)\ (x - x_0).$$

Usually

$$\tilde{u}_{j,i} := \tilde{u} + \varphi_i(u_j - \tilde{u}) \in L^1(\Omega; \mathbb{R}^n).$$



Let us take and fix any x_0 where the function u is approximately differentiable and define

$$\tilde{u}(x) := u(x_0) + \nabla u(x_0)\ (x - x_0).$$

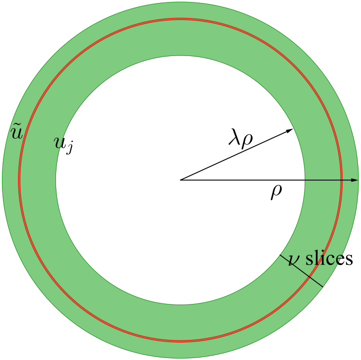
Usually

$$\tilde{u}_{j,i} := \tilde{u} + \varphi_i(u_j - \tilde{u}) \in L^1(\Omega; \mathbb{R}^n).$$

But

$$\operatorname{div} \tilde{u}_{j,i} = (1 - \varphi_i) \operatorname{div} \tilde{u} + \varphi_i \operatorname{div} u_j + \\ + \nabla\varphi_i \cdot (u_j - \tilde{u})$$

and there in no control on the last term L^2.



Let us take and fix any x_0 where the function u is approximately differentiable and define

$$\tilde{u}(x) := u(x_0) + \nabla u(x_0)\ (x - x_0).$$

Usually

$$\tilde{u}_{j,i} := \tilde{u} + \varphi_i(u_j - \tilde{u}) \in L^1(\Omega; \mathbb{R}^n).$$

But

$$\operatorname{div} \tilde{u}_{j,i} = (1 - \varphi_i) \operatorname{div} \tilde{u} + \varphi_i \operatorname{div} u_j + \\ + \nabla\varphi_i \cdot (u_j - \tilde{u})$$

and there in no control on the last term L^2.

$$\zeta_{j,i} := \text{average of } \nabla\varphi_i \cdot (u_j - \tilde{u}) \text{ in } B_i \setminus B_{i-1}.$$

By the result of Bogovskiĭ, there exist $z_{j,i} \in W^{1,q}_0(B_i \setminus \overline{B_{i-1}})$ such that

$$\operatorname{div} z_{j,i} = -\nabla\varphi_i \cdot (u_j - \tilde{u}) + \zeta_{j,i}$$

with

$$\|z_{j,i}\|_{W^{1,q}(B_i \setminus \overline{B_{i-1}})} \leq \frac{C\nu}{(1-\lambda)\rho} \|u_j - \tilde{u}\|_{L^q(B_i \setminus \overline{B_{i-1}})}.$$

lim-inf inequality
Regular points

Now define $u_{j,i} := \tilde{u}_{j,i} + z_{j,i} \in LU(\Omega; \mathbb{R}^n)$. Notice that

$$u_{j,i} - \tilde{u} = \varphi_i(u_j - \tilde{u}) + z_{j,i} \in LU_0(B_\rho(x_0); \mathbb{R}^n).$$

Now define $u_{j,i} := \tilde{u}_{j,i} + z_{j,i} \in LU(\Omega;\mathbb{R}^n)$. Notice that

$$u_{j,i} - \tilde{u} = \varphi_i(u_j - \tilde{u}) + z_{j,i} \in LU_0(B_\rho(x_0);\mathbb{R}^n).$$

Then

$$\begin{aligned} f_{\mathrm{hom}}(\mathfrak{E}u(x_0)) &= \lim_{j\to\infty} \inf_{\varphi\in LU_0(B_\rho(x_0),\mathbb{R}^n)} \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} f(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u(x_0) + \mathfrak{E}\varphi(x))\, dx \\ &\leq \liminf_{j\to\infty} \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} f\big(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\big)\, dx \end{aligned}$$

Now define $u_{j,i} := \tilde{u}_{j,i} + z_{j,i} \in LU(\Omega;\mathbb{R}^n)$. Notice that

$$u_{j,i} - \tilde{u} = \varphi_i(u_j - \tilde{u}) + z_{j,i} \in LU_0(B_\rho(x_0);\mathbb{R}^n).$$

Then

$$\begin{aligned} f_{\text{hom}}(\mathfrak{E}u(x_0)) &= \lim_{j\to\infty} \inf_{\varphi\in LU_0(B_\rho(x_0),\mathbb{R}^n)} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\left(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u(x_0) + \mathfrak{E}\varphi(x)\right)\,dx \\ &\le \liminf_{j\to\infty} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\big(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\big)\,dx \end{aligned}$$

Averaging: $$f_{\text{hom}}(\mathfrak{E}u(x_0)) \le \liminf_{j\to\infty} \frac{1}{\nu}\sum_{i=1}^{\nu} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\big(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\big)\,dx.$$

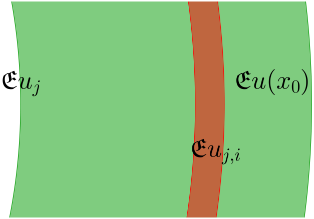
Now define $u_{j,i} := \tilde{u}_{j,i} + z_{j,i} \in LU(\Omega;\mathbb{R}^n)$. Notice that

$$u_{j,i} - \tilde{u} = \varphi_i(u_j - \tilde{u}) + z_{j,i} \in LU_0(B_\rho(x_0);\mathbb{R}^n).$$

Then

$$\begin{aligned} f_{\hom}(\mathfrak{E}u(x_0)) &= \lim_{j\to\infty} \inf_{\varphi\in LU_0(B_\rho(x_0),\mathbb{R}^n)} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u(x_0) + \mathfrak{E}\varphi(x))\,dx \\ &\le \liminf_{j\to\infty} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\big(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\big)\,dx \end{aligned}$$

Averaging: $\quad f_{\hom}(\mathfrak{E}u(x_0)) \le \liminf_{j\to\infty} \frac{1}{\nu}\sum_{i=1}^{\nu} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\big(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\big)\,dx.$



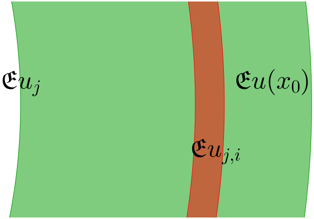
Now define $u_{j,i} := \tilde{u}_{j,i} + z_{j,i} \in LU(\Omega;\mathbb{R}^n)$. Notice that

$$u_{j,i} - \tilde{u} = \varphi_i(u_j - \tilde{u}) + z_{j,i} \in LU_0(B_\rho(x_0);\mathbb{R}^n).$$

Then

$$\begin{aligned} f_{\text{hom}}(\mathfrak{E}u(x_0)) &= \lim_{j\to\infty} \inf_{\varphi\in LU_0(B_\rho(x_0),\mathbb{R}^n)} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u(x_0) + \mathfrak{E}\varphi(x))\,dx \\ &\le \liminf_{j\to\infty} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\big(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\big)\,dx \end{aligned}$$

Averaging: $\quad f_{\text{hom}}(\mathfrak{E}u(x_0)) \le \liminf_{j\to\infty} \frac{1}{\nu}\sum_{i=1}^{\nu} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\big(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\big)\,dx.$



- First term: ✓

lim-inf inequality
Regular points

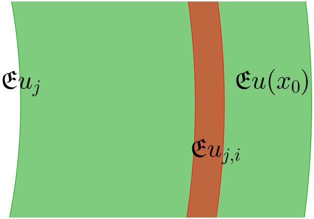
Now define $u_{j,i} := \tilde{u}_{j,i} + z_{j,i} \in LU(\Omega;\mathbb{R}^n)$. Notice that

$$u_{j,i} - \tilde{u} = \varphi_i(u_j - \tilde{u}) + z_{j,i} \in LU_0(B_\rho(x_0);\mathbb{R}^n).$$

Then

$$\begin{aligned} f_{\text{hom}}(\mathfrak{E}u(x_0)) &= \lim_{j\to\infty} \inf_{\varphi\in LU_0(B_\rho(x_0),\mathbb{R}^n)} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u(x_0) + \mathfrak{E}\varphi(x))\,dx \\ &\leq \liminf_{j\to\infty} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\big(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\big)\,dx \end{aligned}$$

Averaging: $\quad f_{\text{hom}}(\mathfrak{E}u(x_0)) \leq \liminf_{j\to\infty} \frac{1}{\nu}\sum_{i=1}^{\nu} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\big(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\big)\,dx.$



- First term: ✓
- Third term: $\lambda \nearrow 1$

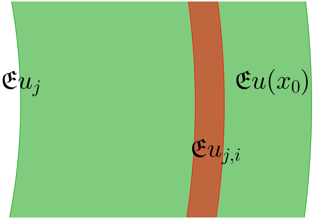
Now define $u_{j,i} := \tilde{u}_{j,i} + z_{j,i} \in LU(\Omega;\mathbb{R}^n)$. Notice that

$$u_{j,i} - \tilde{u} = \varphi_i(u_j - \tilde{u}) + z_{j,i} \in LU_0(B_\rho(x_0);\mathbb{R}^n).$$

Then

$$\begin{aligned} f_{\mathrm{hom}}(\mathfrak{E}u(x_0)) &= \lim_{j\to\infty} \inf_{\varphi\in LU_0(B_\rho(x_0),\mathbb{R}^n)} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\big(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u(x_0) + \mathfrak{E}\varphi(x)\big)\,dx \\ &\le \liminf_{j\to\infty} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\big(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\big)\,dx \end{aligned}$$

Averaging: $$f_{\mathrm{hom}}(\mathfrak{E}u(x_0)) \le \liminf_{j\to\infty} \frac{1}{\nu}\sum_{i=1}^{\nu} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\big(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\big)\,dx.$$



- First term: ✓
- Third term: $\lambda \nearrow 1$
- Second term: L^q-differentiability

We suppose that for every $\eta > 0$ there are

- $\beta_\eta > 0$
- a Carathéodory function $c^\eta : \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ that is $\mathbb{I}^n$-periodic in the first variable and convex in the second,

such that for a.e. $x \in \mathbb{R}^n$ and all $X \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$

$$|f(x,X) - c^\eta(x,X)| \leq \eta(|X_{\mathrm{dev}}| + (\mathrm{tr}\, X)^2) + \beta_\eta.$$

We will refer to this property as *asymptotic convexity*.

Let us notice that for f in our setting we may even suppose

- c^η to be non-negative with $c^\eta(x,0) = 0$ for every $x \in \mathbb{R}^n$,
- $\mathrm{dom}(c^*(x,\_))$ to be closed for a.e. $x \in \mathbb{R}^n$.

We suppose that for every $\eta > 0$ there are

- $\beta_\eta > 0$
- a Carathéodory function $c^\eta : \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ that is $\mathbb{I}^n$-periodic in the first variable and convex in the second,

such that for a.e. $x \in \mathbb{R}^n$ and all $X \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$

$$|f(x,X) - c^\eta(x,X)| \leq \eta(|X_{\mathrm{dev}}| + (\operatorname{tr} X)^2) + \beta_\eta.$$

We will refer to this property as *asymptotic convexity*.

Let us notice that for f in our setting we may even suppose

- c^η to be non-negative with $c^\eta(x,0) = 0$ for every $x \in \mathbb{R}^n$,
- $\mathrm{dom}(c^*(x, \_))$ to be closed for a.e. $x \in \mathbb{R}^n$.

[Demengel, Qi 90]: For such convex function c

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{C}_\varepsilon = \mathcal{C}_{\mathrm{hom}} \quad \text{with} \quad \mathcal{C}_{\mathrm{hom}}(u) := \begin{cases} \int_\Omega c_{\mathrm{hom}}\big(Eu(x)\big), & u \in U(\Omega;\mathbb{R}^n), \\ \infty, & \text{else}, \end{cases}$$

where $\quad c_{\mathrm{hom}}(X) = \inf\limits_{\varphi \in LU_{\mathrm{per}}(\mathbb{I}^n;\mathbb{R}^n)} \int_{\mathbb{I}^n} c\big(x, X + \mathfrak{E}\varphi(x)\big)\, dx \quad$ and

$$c_{\mathrm{hom}}\big(Eu(x)\big) = c_{\mathrm{hom}}\big(\mathfrak{E}u(x)\big)\, dx + (c_{\mathrm{hom}})^{\#}\left(\frac{dE^s u}{d|E^s u|}(x)\right) d|E^s u|(x).$$

lim-inf inequality
Singular points

We have

$$\mu_j := f(\tfrac{\cdot}{\varepsilon_j}, \mathfrak{E}u_j(\cdot))\mathcal{L}^n.$$

We may suppose

- $\lim_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j)$ equals the lim inf above with all $u_j \in LU(\Omega; \mathbb{R}^n)$,
- $u_j \to u$ in $L^q(\Omega; \mathbb{R}^n)$ due to the lower bound on f and since LU is compactly embedded in L^q,
- $\mu_j \overset{*}{\rightharpoonup} \mu$ in $M(\Omega; \mathbb{R}^n)$,
- $(|\mathfrak{E}_{\mathrm{dev}} u_j| + (\operatorname{div} u_j)^2)\mathcal{L}^n \overset{*}{\rightharpoonup} \sigma$ in $M(\Omega)$.

We have

$$\mu_j := f(\tfrac{\cdot}{\varepsilon_j}, \mathfrak{E}u_j(\cdot))\mathcal{L}^n.$$

We may suppose

- $\lim_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j)$ equals the lim inf above with all $u_j \in LU(\Omega;\mathbb{R}^n)$,
- $u_j \to u$ in $L^q(\Omega;\mathbb{R}^n)$ due to the lower bound on f and since LU is compactly embedded in L^q,
- $\mu_j \overset{*}{\rightharpoonup} \mu$ in $M(\Omega;\mathbb{R}^n)$,
- $(|\mathfrak{E}_{\text{dev}} u_j| + (\operatorname{div} u_j)^2)\mathcal{L}^n \overset{*}{\rightharpoonup} \sigma$ in $M(\Omega)$.

For each $\eta > 0$

$$\begin{aligned} f(x,X) &\geq c^\eta(x,X) - \eta(|X_{\text{dev}}| + (\operatorname{tr} X)^2) - \beta_\eta \\ \mu &\geq c^\eta_{\text{hom}}(Eu) - \eta\,\sigma - \beta_\eta \mathcal{L}^n \\ \mu^s &\geq (c^\eta_{\text{hom}})^\#(\tfrac{dE^s u}{d|E^s u|})|E^s u| - \eta\,\sigma^s \end{aligned}$$

Since

$$\lim_{\eta\to 0} (c^\eta_{\text{hom}})^\#(X) = (f_{\text{hom}})^\#(X),$$

and

$$\mu^s \geq (f_{\text{hom}})^\#\big(\tfrac{dE^s u}{d|E^s u|}\big)|E^s u|.$$

Theorem

Let us have a Carathéodory function $f : \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\text{sym}} \to \mathbb{R}$ *that*

- *is* $\mathbb{I}^n$*-periodic in the first variable,*
- *has Hencky plasticity growth.*

Let us denote

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_\Omega f\left(\frac{x}{\varepsilon}, \mathfrak{E}u(x)\right)\, dx, & u \in LU(\Omega;\mathbb{R}^n), \\ \infty, & \text{else}, \end{cases}$$

and

$$\mathcal{F}_{\text{hom}}(u) := \begin{cases} \int_\Omega f_{\text{hom}}\left(\mathfrak{E}u(x)\right) dx + \int_\Omega (f_{\text{hom}})^{\#}\left(\frac{dE^s u}{d|E^s u|}(x)\right) d|E^s u|(x), & u \in U(\Omega;\mathbb{R}^n), \\ \infty, & \text{else}. \end{cases}$$

Then

$$\Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon \leq \mathcal{F}_{\text{hom}},$$

while for $u \in LU(\Omega;\mathbb{R}^n)$ *even*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u) = \mathcal{F}_{\text{hom}}(u).$$

The latter holds for all $u \in L^1(\Omega;\mathbb{R}^n)$ *if* f *is asymptotically convex.*

Theorem

With assumptions and denotations as above, including the asymptotic convexity, the following diagrams commute:

$$
\begin{array}{ccc}
\mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\text{pt. falling}} & \mathcal{F}_\varepsilon^{(0)} \\
\Big\downarrow \Gamma & & \Big\downarrow \Gamma \\
\mathcal{F}_{\mathrm{hom}}^{(\delta)} & \xrightarrow[\Gamma]{} & \mathcal{F}_{\mathrm{hom}}
\end{array}
\qquad \textit{and} \qquad
\begin{array}{ccc}
\mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\Gamma} & \operatorname{lsc} \mathcal{F}_\varepsilon^{(0)} = \operatorname{lsc} \mathcal{F}_\varepsilon \\
\Big\downarrow \Gamma & & \Big\downarrow \Gamma \\
\mathcal{F}_{\mathrm{hom}}^{(\delta)} & \xrightarrow[\Gamma]{} & \mathcal{F}_{\mathrm{hom}}
\end{array}
$$

All Γ-limits are with respect to the L^1-norm.

Thank you for your attention