Non-trivial pinning threshold for an evolution equation involving long range interactions

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joint work with Patrick Dondl

Workshop: "New trends in the variational modeling of failure phenomena"
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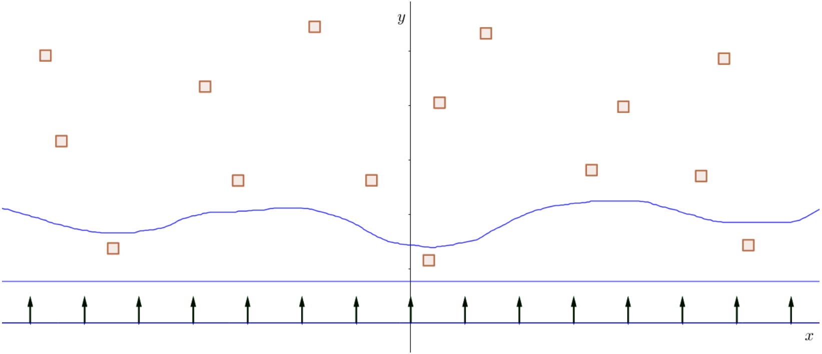
Moving of interfaces



An interface moves through a media driven by a constant force F upwards. In the media, there are obstacles that act with a force $f(x, y)$ downwards.

We suppose the interface to move accroding to the curvature flow. Thus

$$v_n = \varkappa - f + F.$$

Starting with a flat interface $u = 0$ at time 0, we suppose the interface to be given by a graph at all times. Assuming the gradient to be small, we arrive at

$$\frac{\partial u}{\partial t}(x,t) = \triangle u(x,t) - f(x, u(x,t)) + F$$

Deterministic periodic setting

Theorem (Dirr, Yip)

If the array of obstacles is deterministic and 1-perodic (positions and strengths),

Dirr, N.; Yip, N. K. Pinning and de-pinning phenomena in front propagation in heterogeneous media. Interfaces Free Bound. 8 (2006), no. 1, 79–109.

Deterministic periodic setting

Theorem (Dirr, Yip)

If the array of obstacles is deterministic and 1-perodic (positions and strengths), then there exists $F_* > 0$ such that

Pinning

for any $0 \leq F \leq F_*$ there exists a stationary solution $U_F > 0$, and because of the comparison principle the interface "gets pinned" under the graph of U_F.

Depinning

for any $F > F_*$ there exists an unique $T_F > 0$ and a spatially 1-periodic solution U_F with

$$U_F(\_, t + T_F) = U_F(\_, t) + 1.$$

Dirr, N.; Yip, N. K. Pinning and de-pinning phenomena in front propagation in heterogeneous media. Interfaces Free Bound. 8 (2006), no. 1, 79–109.

Quenched Edwards-Wilkinson model

We suppose that the obstacles have the same shape and (time-independent) random positions and strenghts. More precisely, the moving is determined by the equation

$$\frac{\partial u(x,t,\omega)}{\partial t} = \triangle u(x,t,\omega) - f(x, u(x,t,\omega), \omega) + F$$

with the force of the obstacle field being the random function

$$f(x,y,\omega) = \sum_i f_i(\omega)\varphi(x - x_i(\omega), y - y_i(\omega)).$$

We assume

Condition 1 (Shape of obstacles)

There exist $r_0, r_1 > 0$ with $r_1 > \sqrt{n} r_0$ so that

$$\varphi(x,y) \geq 1 \text{ for } \|(x,y)\|_\infty \leq r_0 \quad \text{and} \quad \varphi(x,y) = 0 \text{ for } \|(x,y)\| \geq r_1.$$

Condition 2 (Obstacle positions and strengths)

The random distribution of obstacle sites $\{(x_i, y_i)\}_{i\in\mathbb{N}} \subset \mathbb{R}^n \times [r_1, \infty)$ and strengths $\{f_i\}_{i\in\mathbb{N}} \subset [0, \infty)$ satisfy:

- $\{(x_i, y_i)\}_{i\in\mathbb{N}}$ are distributed according to an $(n+1)$-dimensional Poisson point process on $\mathbb{R}^n \times [r_1, \infty)$ with intensity $\lambda > 0$,
- $\{f_i\}_{i\in\mathbb{N}}$ are independent and identically distributed strictly positive random variables that are independent of $\{(x_i, y_i)\}_{i\in\mathbb{N}}$.

Pinning result

Theorem (Dirr, Dondl, Scheutzow)

If Conditions 2.1 and 2.2 are satisfied, then there exists $F_* > 0$ *and a non-negative* $v : \mathbb{R}^n \times \Omega \to [0, \infty)$ *so that*

$$0 \geq \triangle v(x, \omega) - f(x, v(x, \omega), \omega) + F_*$$

almost surely. Hence, for $F \leq F_*$ *any solution for Quenched Edwards-Wilkinson model with trivial initial condition gets pinned.*

Dirr, N.; Dondl, P. W.; Scheutzow, M. Pinning of interfaces in random media. Interfaces Free Bound. 13 (2011), no. 3, 411–421

Dondl, P. W.; Scheutzow, M. Positive speed of propagation in a semilinear parabolic interface model with unbounded random coefficients. Netw. Heterog. Media 7 (2012), no. 1, 137–150

Long range interactions

We will explore the same question for the evolution problem

$$\frac{\partial u(x,t,\omega)}{\partial t} = -(-\triangle)^s u(x,t,\omega) - f(x, u(x,t,\omega), \omega) + F, \quad u(x,0,\omega) = 0$$

Question

Does there exist a $F_* > 0$ and a non-negative $v : \mathbb{R}^n \times \Omega \to [0, \infty)$ so that

$$0 \geq -(-\triangle)^s v(x,\omega) - f(x, v(x,\omega), \omega) + F_*$$

almost surely?

Our goal is to prove

Answer

There exists such F_* and v, and the function $u(x,t,\omega) := \min\{v(x,\omega), F_* t\}$ is a viscosity supersolution of the evolution problem for $F \leq F_*$ almost surely.

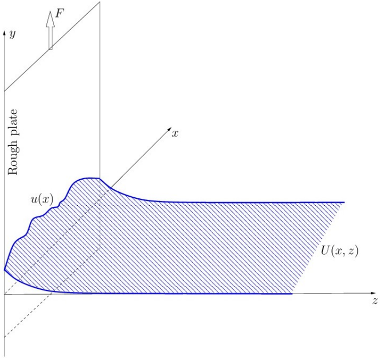
For one dimensional case this has been already done.

Throm, S. Pinning of interfaces in a random elastic medium. Diplom-Arbeit.

Dondl, P. W.; Scheutzow, M.; Throm, S. Pinning of interfaces in a random elastic medium and logarithmic lattice embeddings in percolation. Proc. Roy. Soc. Edinburgh Sect. A 145 (2015), no. 3

Examples I

- Wetting line when pulling sandpaper out of water ($s = \frac{1}{2}$)



[by Sebastian Throm]

Dondl, P. W.; Scheutzow, M.; Throm, S. Pinning of interfaces in a random elastic medium and logarithmic lattice embeddings in percolation. Proc. Roy. Soc. Edinburgh Sect. A 145 (2015), no. 3

Ertaş, D.; Kardar, M. Critical dynamics of contact line depinning, Phys. Rev. E 49 (1994)

Examples II

- Motion of crack fronts in heterogeneous media

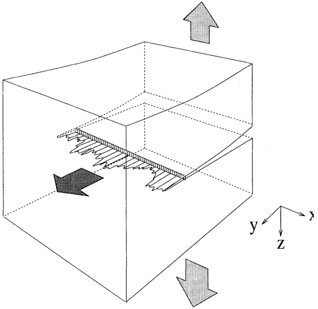


FIG. 1. Geometry of the interfacial crack modeled in this letter. The crack front is planar $z = 0$. It is aligned along the x direction, and it propagates along y. A pure tensile stress is applied at infinity, so that the crack opens in mode I.

[from the first article below]

Gao, H.; Rice J. R. A First-Order Perturbation Analysis of Crack Trapping by Arrays of Obstacles, J. Appl. Mech 56 (4) (1989), 828-836

Schmittbuhl, J.; Roux, S.; Vilotte, J.-P.; Måløy, K. J. Interfacial Crack Pinning: Effect of Nonlocal Interactions, Phys. Rev. Lett. 74, 1787 (1995)

Ramanathan, S.; Ertaş, D.; Fisher, D. Quasistatic Crack Propagation in Heterogeneous Media Phys. Rev. Lett. 79, 873 (1997)

Examples III

- fractional diffusion equations

  Considering the probabilistic view-point of diffusion, here we allow for arbitrary big jumps (with corresponding weighted probabilities).
- equations describing the dynamics of fluids in porous media
- dislocations dynamics that is used to explain the plastic deformation of crystals

Examples IV

- study of displacive solid-solid phase transformations (2-D example)

  Dondl, P. W.; Bhattacharya, K. Effective behavior of an interface propagating through a periodic elastic medium. Interfaces Free Bound. 18 (2016), no. 1

  They show that a nearly flat interface is given by the graph of the function g which evolves according to the equation

$$\frac{\partial g}{\partial t}(x,t) = -(-\triangle)^{1/2} g(x,t) + \varphi(x, g(x)) + F.$$

Then

Theorem (Bhattacharya, Dondl)

If the array of obstacles is deterministic and 1-perodic (positions and strengths), then there exists $F_* > 0$ such that

Pinning	Depinning
for any $0 \leq F \leq F_*$	*for any* $F > F_*$

Fractional Laplace operator

(At least) for $u \in \mathcal{S}$, we may define fractional Laplace operator for $s \in (0,1)$

- with singular integral

$$(-\triangle)^s u(x) := C(n,s)\ P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\ dy$$

  where

$$C(n,s)^{-1} := \int_{\mathbb{R}^n} \frac{1-\cos\zeta_1}{|\zeta|^{n+2s}}\ d\zeta,$$

- via Fourier-Transform

$$\mathcal{F}\Big((-\triangle)^s u\Big)(\xi) = |\xi|^{2s}(\mathcal{F}u)(\xi),$$

- as a regular integral

$$-(-\triangle)^s u(x) = \frac{C(n,s)}{2} \int_{\mathbb{R}^n} \frac{u(x+h)+u(x-h)-2u(x)}{|h|^{n+2s}}\ dh.$$

Di Nezza, E.; Palatucci, G.; Valdinoci, E. Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. 136 (2012), no. 5

Kwaśnicki, M. Ten equivalent definitions of the fractional Laplace operator. Fract. Calc. Appl. Anal. 20 (2017), no. 1

Extension property

If for a given $f : \mathbb{R}^n \to \mathbb{R}$, we solve

$$\begin{aligned} u(x,0) &= f(x) && \text{for } x \in \mathbb{R}^n, \\ \triangle u(x,y) &= 0 && \text{for } x \in \mathbb{R}^n,\ y > 0, \end{aligned}$$

then $-\frac{\partial}{\partial y} u(x,0) = C(-\triangle)^{1/2} f(x)$.

For other $s \in (0,1)$: If

$$\begin{aligned} u(x,0) &= f(x) && \text{for } x \in \mathbb{R}^n, \\ \operatorname{div}\left(y^{1-2s} \nabla u(x,y)\right) &= 0 && \text{for } x \in \mathbb{R}^n,\ y > 0, \end{aligned}$$

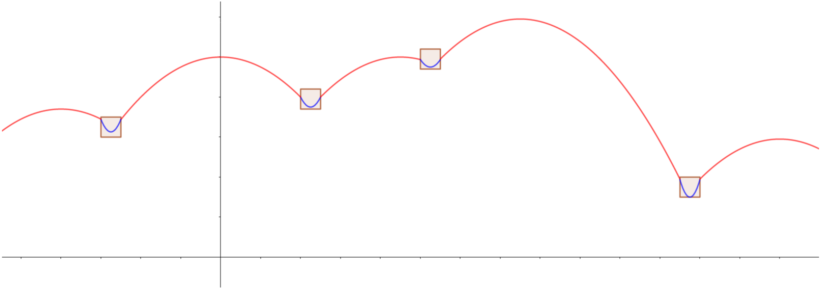
then

$$\lim_{y \searrow 0} \left(y^{1-2s} \frac{\partial u}{\partial y}(x,y) \right) = C(-\triangle)^s f(x).$$

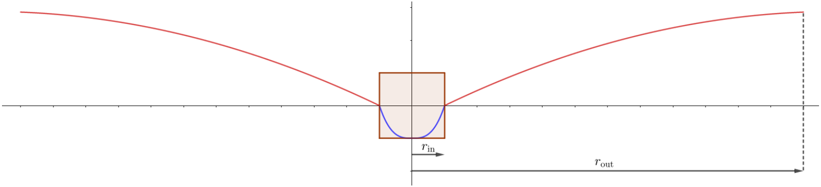
Caffarelli, L.; Silvestre, L. An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007)

Idea for QEW model

For QEW model, we construct a supersolution



by finding a local radial symmetric solution and lifting it:



$$\triangle v_{in} = F_{in} \left(\frac{r}{r_{in}}\right)^m \quad \text{and} \quad \triangle v_{out} = -F_{out}, \quad v'_{out}(r_{out}) = 0.$$

Dirichlet problem for fractional Laplacian

What is the right notion of Dirichlet problem for the fractional Laplacian, e.g. with zero boundary condition? The fact is that $(-\triangle)^s$ is non-local!

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We consider

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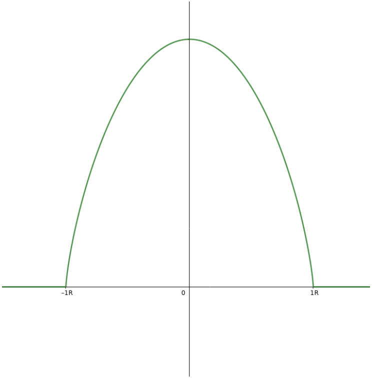
The problem

$$\begin{aligned} (-\triangle)^s v(x) = 1 \quad & \text{for } |x| < R, \\ v(x) = 0 \quad & \text{for } |x| \geq R \end{aligned}$$

has the solution v with

$$v(x) = \frac{\Gamma(\frac{n}{2})}{2^{2s}\Gamma(\frac{n}{2}+s)\Gamma(1+s)}(R^2 - |x|^2)^s$$

for $|x| < R$.



Local radial solution

Let us look at the problem

$$\begin{aligned} -(-\triangle)^s v(x) + s(x) &= 0 \quad \text{for } |x| < R, \\ v(x) &= 0 \quad \text{for } |x| \geq R. \end{aligned}$$

We may compute

$$v(x) = \int_{B_R(0)} G_{n,s}(x,y) s(y) \, dy$$

with $G_{n,s}$ being the Green function

$$G_{n,s}(x,y) = \frac{\Gamma(\frac{n}{2})}{2^{2s}\pi^{1/n}\Gamma(s)^2} \frac{1}{|x-y|^{n-2s}} \Phi_{n,s}(x,y)$$

where

$$\Phi_{n,s}(x,y) = \int_0^{\zeta} \frac{1}{w^{1-s}} \frac{1}{(1+w)^{n/2}} \, dw \quad \text{and} \quad \zeta = \frac{1}{R^2} \frac{(R^2-|x|^2)(R^2-|y|^2)}{|x-y|^2}.$$

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For given $F_1, F_2 > 0$, the solution

$$\begin{aligned} -(-\triangle)^s v(x) &= \begin{cases} F_1, & \text{if } |x| < r_0, \\ -F_2, & \text{if } r_0 < |x| < R, \end{cases} \\ v(x) &= 0 \quad \text{if } |x| \geq R, \end{aligned}$$

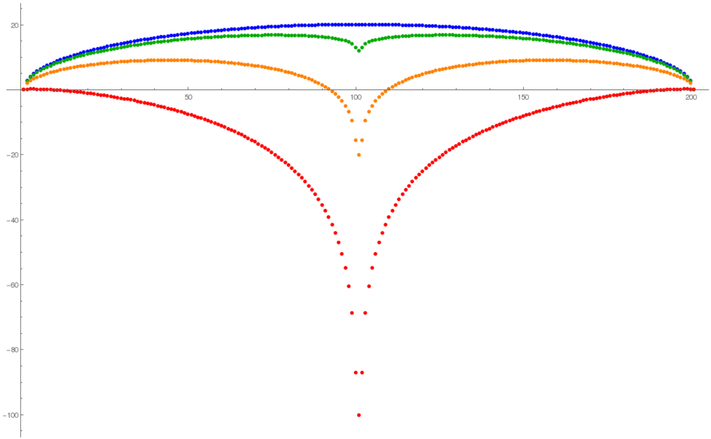
is therefore given by

$$v(x) = F_2 \int_{B_R(0)} G_{n,s}(x,y) \, dy - (F_1 + F_2) \int_{B_{r_0}(0)} G_{n,s}(x,y) \, dy.$$

Local radial solution

Let $r_0 = qR$ with $q \in (0,1)$. We would like to explore the interplay of these two integrals, and find an appropriate scaling for F_1, F_2, q in order for the solution the behave as desired. Pictures for

$$R = 10,\ q = 0{,}01,\ s = \frac{1}{2},\ \frac{F_1}{F_2} = 0, 10, 50, 150$$



Flat supersolution

The goal: to assess when $v \leq 0$ and v radially increasing. By estimating the integrals, we arrive at

$$\frac{F_1 + F_2}{F_2} \geq \frac{n}{2s} \frac{(1+q)^n}{(1-q^2)^s} \frac{1}{q^n}.$$

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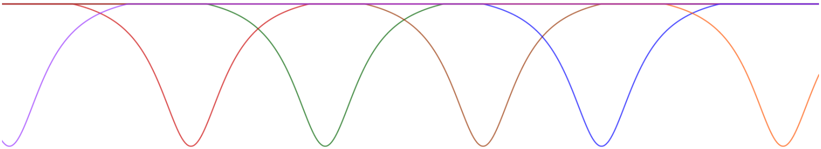
Then we may define

$$v_{\text{flat}} : \mathbb{R}^n \times \Omega \to \mathbb{R}, \quad v_{\text{flat}}(x, \omega) := \min_{\text{some obstacles}} v_{\text{local}}(x - x_{\text{obstacle}}).$$



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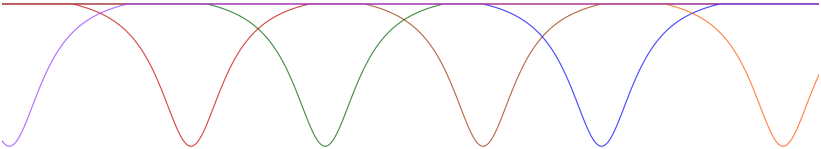


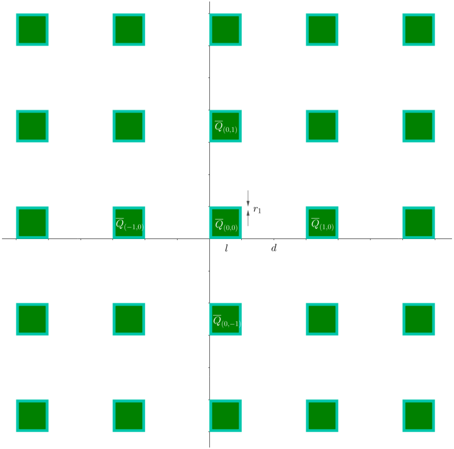
Since

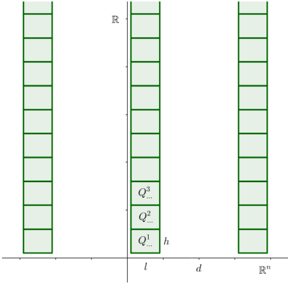
$$-(-\triangle)^s u(x) = C \; P.V. \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|y - x|^{n+2s}} \; dy,$$

we have the right inequality

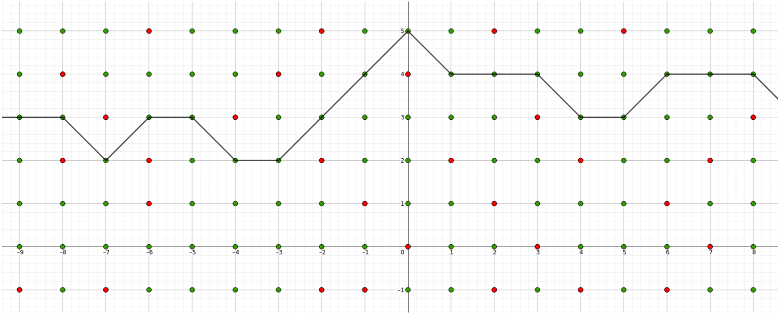
$$-(-\triangle)^s v_{\text{flat}}(x) = C \; P.V. \int_{\mathbb{R}^n} \frac{v_{\text{flat}}(y) - v_{\text{flat}}(x)}{|y - x|^{n+2s}} \; dy \leq -(-\triangle)^s v_{\text{local}}(x - x_a(\omega)).$$

Partition of the space





Percolation



Theorem (Dondl, Scheutzow, Throm)

Suppose $z \in \mathbb{Z}^{n+1}$ is open with probability $p \in (0,1)$ and closed otherwise, with different sites receiving independent states. For every nondecreasing function $H : \mathbb{N} \to \mathbb{N}$ with

$$\liminf_{k \to \infty} \frac{H(k)}{\log k} > 0,$$

there exists $p_H = p_H(n) \in (0,1)$ such that for every $p \in (p_H, 1)$ there exists a.s. a (random) function $L : \mathbb{Z}^n \to \mathbb{N}$ with the following properties:

- *For each $a \in \mathbb{Z}^n$, the site $(a, L(a)) \in \mathbb{Z}^{n+1}$ is open.*
- *For any $a, b \in \mathbb{Z}^n$, $a \neq b$, it holds $|L(a) - L(b)| \leq H(\|a - b\|_1)$.*

Percolation result applied

Take any $S > 0$ such that $P(\text{obstacle strength} \geq S) =: p_S > 0$.

Probability that in a cuboid with volume V there is a centre of an obstacle with strength at least S is $1 - \exp(-\lambda V p_S)$.

Percolation result applied

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Thus, if $1 - \exp(-\lambda h(l - 2r_1)^n p_S) > p_H$, i.e.

$$\boxed{h(l - 2r_1)^n > -\frac{1}{\lambda p_S} \log(1 - p_H)}$$

then the Percolation theorem is applicable.

Hence, there exists a.s. a random function $L : \mathbb{Z}^n \times \Omega \to \mathbb{N}$ such that

- for each $a \in \mathbb{Z}^n$, the cuboid $Q_a^{L(a)}$ contains a center of an obstacle (x_a, y_a) with strength at least S,
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Let $R \geq \sqrt{n}\left(l + \frac{d}{2} - r_1\right)$. We define the flat supersolution (with given F_1, F_2, q)

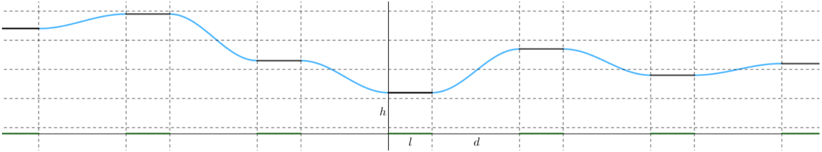
$$v_{\text{flat}} : \mathbb{R}^n \times \Omega \to \mathbb{R}, \quad v_{\text{flat}}(x, \omega) := \min_{a \in \mathbb{Z}^n} v_{\text{local}}(x - x_a(\omega)).$$

then v_{flat} satisfies a.s. in the sense of distributions (and in the sense of viscosity solutions)

$$0 \geq -(-\triangle)^s v_{\text{flat}}(x) - \sum_{a \in \mathbb{Z}^n} S\varphi(x - x_a, v_{\text{flat}}(x)) + F$$

for all $F \leq \min\{S - F_1, F_2\}$.

Lifting function



Proposition

Let $h, d, l > 0$ and $s \in (0,1)$. Suppose $y : \mathbb{Z}^n \to \mathbb{R}$ has the following property

$$|y_a - y_b| \leq 2h\|a-b\|_1^\alpha$$

for some $0 < \alpha < 2s$. Then there exist a smooth function $u_{\text{lift}} : \mathbb{R}^n \to \mathbb{R}$ and constants C_0, C_1, C_2 depending only on n, s, α such that:

- $u_{\text{lift}}(x) = y_a$ *if* $x \in Q_a$ *for some* $a \in \mathbb{Z}^n$,
- $\|D^2 u_{\text{lift}}\|_\infty \leq C_0 \frac{h}{d^2}$,
- $\|(-\triangle)^s u_{\text{lift}}\|_\infty \leq C_1 (d+l)^{2-2s} \frac{h}{d^2} + C_2 \frac{h}{(d+l)^{2s}}$.

Right scaling

Let us fix $S > 0$ such that $P(\text{obstacle strength} \geq S) = p_S > 0$. We must be able to choose appropriate h, l, d, R, q, F_1, F_2:

1. **We must have enough obstacles for the percolation result:**

$$\boxed{h(l - 2r_1)^n > -\frac{1}{\lambda p_S} \log(1 - p_H)}$$

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2. **The local solution must be non-positive and radially increasing:**

$$\boxed{\frac{F_1 + F_2}{F_2} \geq \frac{n}{2s} \frac{(1+q)^n}{(1-q^2)^s} \frac{1}{q^n}}$$

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3. **The local solution must stay inside an obstacle:**
We scale the local solutions so that $|u_{\text{local}}| < r_0$. This will hold if

$$\boxed{\frac{\pi^{n/2}}{2^{2s}\pi^{1/n}\Gamma(s)^2 s^2 (\frac{n}{2} - s)} F_1 r_0^{2s} \leq r_0}$$

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$$h(l - 2r_1)^n > -\frac{1}{\lambda p_S} \log(1 - p_H)$$

2. **The local solution must be non-positive and radially increasing:**

$$\frac{F_1 + F_2}{F_2} \geq \frac{n}{2s} \frac{(1+q)^n}{(1-q^2)^s} \frac{1}{q^n}$$

3. **The local solution must stay inside an obstacle:**
We scale the local solutions so that $|u_{\text{local}}| < r_0$. This will hold if

$$\frac{\pi^{n/2}}{2^{2s}\pi^{1/n}\Gamma(s)^2 s^2 (\frac{n}{2} - s)} F_1 r_0^{2s} \leq r_0$$

4. **The cost of lifting:**
We have to lift the flat supersolution to the obstacles. Suppose we "spend" $F^* := \frac{1}{2}\min\{S - F_1, F_2\}$ on it. Then we must achieve $|(-\triangle)^s u_{\text{lift}}| \leq F^*$, or

$$C_1(d+l)^{2-2s}\frac{h}{d^2} + C_2 \frac{h}{(d+l)^{2s}} \leq \frac{1}{2}\min\{S - F_1, F_2\}$$

Right scaling

We choose $d := l$, and let us take for the sake of simplicity $R = 2l\sqrt{n} = \frac{1}{q} r_0$.

Since we will choose $l > 4r_1$ (and therefore $l - 2r_1 \geq \frac{l}{2}$), the first will be fulfilled if $hl^n > -2^n \frac{1}{\lambda p_S} \log(1 - p_H)$. Employing $2l = \frac{r_0}{q\sqrt{n}}$, we arrive at

$$\boxed{\frac{h}{q^n} > A_1 := -\frac{2^{2n} n^{n/2}}{r_0^n} \frac{1}{\lambda p_S} \log(1 - p_H)}$$

The second will surely be fulfilled if (with assuming just $q < \frac{1}{2}$)

$$\boxed{\frac{F_1}{F_2} q^n \geq A_2 := \frac{n}{2s} \frac{(\frac{3}{2})^n}{(\frac{3}{4})^s}}$$

For the third, we must get

$$\boxed{F_1 \leq A_3 := r_0^{1-2s} \frac{2^{2s} \pi^{1/n} \Gamma(s)^2 s^2 (\frac{n}{2} - s)}{\pi^{n/2}}}$$

As for the last, it should hold $(4C_1 + C_2) \frac{h}{(2l)^{2s}} \leq \frac{1}{2} \min\{S - F_1, F_2\}$, or, with $A_4 := \frac{4C_1 + C_2}{r_0^{2s}}$,

$$\boxed{A_4 h q^{2s} \leq \frac{1}{2} \min\{S - F_1, F_2\}}$$

Right scaling

The question is if for given A_i and S, there exist such $q \in (0,1)$ and $F_1, F_2, h > 0$. Let q be free and set

$$F_1 := \min\{A_3, \tfrac{S}{2}\}, \quad F_2 := \frac{\min\{A_3, \frac{S}{2}\}}{A_2} q^n.$$

Thus, the second and the third inequality are fulfilled. Obviously, for q small enough

$$\min\{S - F_1, F_2\} = F_2 = \frac{\min\{A_3, \frac{S}{2}\}}{A_2} q^n.$$

So the two remaining inequalities read

$$\boxed{\frac{h}{q^n} > A_1, \quad \frac{h}{q^n} q^{2s} \leq A_5}$$

with $A_5 := \frac{\min\{A_3, \frac{S}{2}\}}{A_2 A_4}$. Hence, we first fix the quotient $\frac{h}{q^n}$, and then choose q sufficiently small. Thus, all the inequalities are fulfilled, and the pinning occurs at least for $F \leq F^*$.

Thank you for your attention