Closure and commutability results for Γ-limts

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joint work with Bernd Schmidt (Universität Augsburg)

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Bernd Schmidt, M. J. Closure and commutability results for Γ-limits and the geometric linearization and homogenization of multiwell energy functionals. SIAM J. Math. Anal. 46 (2014), no. 4, 2525-2553.

Plan of the talk

Introduction and motivation

Abstract results and extensions

Case $p = 1$

Γ-convergence

Definition

Let $\{\mathcal{F}_j : M \to [-\infty, \infty]\}_{j\in\mathbb{N}}$ be a sequence of functionals on a metric space (M, d). Then $\{\mathcal{F}_j\}_{j\in\mathbb{N}}$ Γ-converges at $x \in M$ to some $\mu \in [-\infty, \infty]$ if the following conditions are satisfied:

- (liminf-inequality) If $x_j \to x$ in M, then
$$\liminf_{j\to\infty} \mathcal{F}_j(x_j) \geq \mu.$$
- (recovery sequence) There exists a sequence $x_j \to x$ in M such that
$$\lim_{j\to\infty} \mathcal{F}_j(x_j) = \mu.$$

Denotation:
$$\mu = \Gamma(d)\text{-}\lim_{j\to\infty} \mathcal{F}_j(x).$$

We say that $\{\mathcal{F}_j\}_{j\in\mathbb{N}}$ Γ-converges to some functional $\mathcal{F}_\infty$, if it Γ-converges to $\mathcal{F}_\infty(x)$ at every $x \in M$.

Fundamental properties

Γ-limits are always lower semicontinuous.

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Theorem

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- *there exists a compact set* $K \subset M$ *such that for all* $j \in \mathbb{N}$

$$\inf_{x\in K} \mathcal{F}_j(x) = \inf_{x\in M} \mathcal{F}_j(x).$$

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Then

$$\exists \min_{x\in M} \mathcal{F}_\infty(x) = \lim_{j\to\infty} \inf_{x\in M} \mathcal{F}_j(x).$$

Moreover, if $\{x_j\}_{j\in\mathbb{N}}$ *is a precompact sequence such that*

$$\lim_{j\to\infty} \mathcal{F}_j(x_j) = \lim_{j\to\infty} \inf_{x\in M} \mathcal{F}_j(x),$$

then every limit of a subsequence of $\{x_j\}_{j\in\mathbb{N}}$ *is a minimum point for* $\mathcal{F}_\infty$.

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Theorem (Urysohn property)

Take $\lambda \in [-\infty, \infty]$ *and* $x \in M$. *Then*

$$\lambda = \Gamma(d)\text{-}\lim_{j\to\infty} \mathcal{F}_j(x) \iff \forall \{j_k\}_{k\in\mathbb{N}} \subset \mathbb{N}\ \exists \{k_l\}_{l\in\mathbb{N}} \subset \mathbb{N} : \lambda = \Gamma(d)\text{-}\lim_{l\to\infty} \mathcal{F}_{j_{k_l}}(x)$$

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- $\{\mathcal{F}_j\}_{j\in\mathbb{N}}$ Γ-converges if and only if $\{\operatorname{lsc} \mathcal{F}_j\}_{j\in\mathbb{N}}$ Γ-converges (and Γ-limits then coincide).

Linear elasticity

Equilibria of a hyperelastic material (with given boundary values) can be viewed upon as minimizers of $\int_\Omega W\left(x, \nabla v(x)\right)\, dx$ with W being the stored-energy function. Let us suppose:

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If the displacements are small, i.e. $v(x) = x + \delta u(x)$, then

$$\frac{1}{\delta^2} W(x, I + \delta Y) \approx \frac{1}{2} \partial_Y^2 W(x, I)[Y_{\text{sym}}, Y_{\text{sym}}].$$

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$$\mathcal{E}^{(0)}(u) = \begin{cases} \frac{1}{2} \int_\Omega \mathbf{A}(x)[\mathfrak{E}u(x), \mathfrak{E}u(x)]\, dx, & u \in u_0 + W_0^{1,2}(\Omega; \mathbb{R}^n), \\ \infty, & \text{else on } L^2(\Omega; \mathbb{R}^n), \end{cases}$$

is a good approximation of the original energy

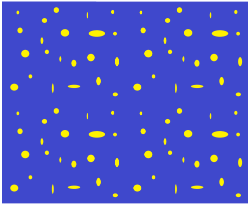
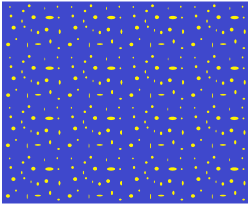
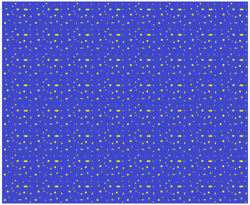
$$\mathcal{E}^{(\delta)}(u) := \begin{cases} \frac{1}{\delta^2} \int_\Omega W(x, I + \delta \nabla u(x))\, dx, & u \in u_0 + W_0^{1,2}(\Omega; \mathbb{R}^n), \\ \infty, & \text{else on } L^2(\Omega; \mathbb{R}^n), \end{cases}$$

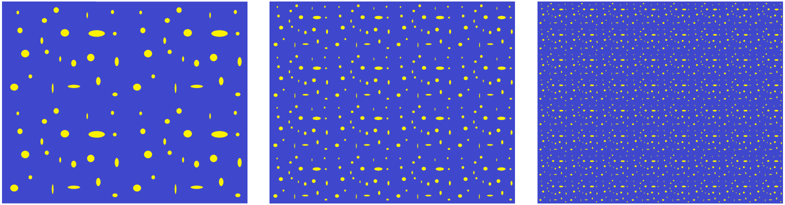
since

$$\Gamma(L^2)\text{-} \lim_{\delta \to 0} \mathcal{E}^{(\delta)} = \mathcal{E}^{(0)}.$$

Dal Maso, Negri, Percivale (2002)

Homogenization (and relaxation)



Suppose $W : \mathbb{R}^n \times \mathbb{R}^{m\times n} \to \mathbb{R}$

- is $\mathbb{I}^n$-periodic in the first variable (with $\mathbb{I} := (0,1)$),
- $\alpha|X|^p \leq W(x,X) \leq \beta\big(|X|^p + 1\big)$.

Under these conditions the family of functionals $\mathcal{E}_\varepsilon$, $\varepsilon > 0$, given by

$$\mathcal{E}_\varepsilon(y) := \begin{cases} \int_\Omega W(\frac{x}{\varepsilon}, \nabla y(x))\, dx, & y \in W^{1,p}(\Omega;\mathbb{R}^m), \\ \infty, & \text{else on } L^p(\Omega;\mathbb{R}^m), \end{cases}$$

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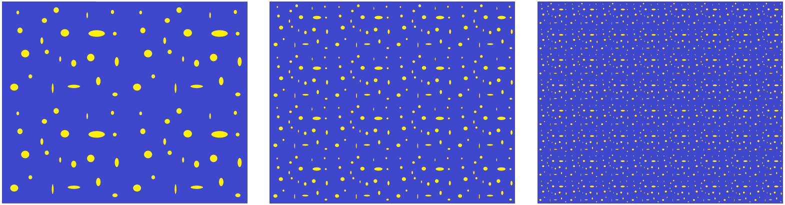

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$\Gamma(L^p)$- converges to

$$\mathcal{E}_{\mathrm{hom}}(y) = \begin{cases} \int_\Omega W_{\mathrm{hom}}(\nabla y(x))\, dx, & y \in W^{1,p}(\Omega;\mathbb{R}^m), \\ \infty, & \text{else on } L^p(\Omega;\mathbb{R}^m). \end{cases}$$

The homogenized stored-energy function is given by

$$W_{\mathrm{hom}}(X) = \inf_{k\in\mathbb{N}} \inf \left\{ \frac{1}{k^n} \int_{k\mathbb{I}^n} W(x, X + \nabla\varphi(x))\, dx : \varphi \in W_0^{1,p}(k\mathbb{I}^n;\mathbb{R}^m) \right\}.$$

Braides (1985), Müller (1987)

Linearization + homogenization = ?

If may have a material with fine periodic structure and small displacements. In this case:

$$\frac{1}{\delta^2}\int_\Omega W\left(\frac{x}{\varepsilon}, I+\delta\nabla u(x)\right)\,dx \xrightarrow{\Gamma} \int_\Omega \mathbf{A}\left(\frac{x}{\varepsilon}\right)[\mathfrak{E}u(x),\mathfrak{E}u(x)]\,dx$$

$$\Big\downarrow \Gamma$$

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\end{array}$$

- linearization ✓
- homogenization ✓
- commutability: Müller, Neukamm (2011)

Geometric linearization in the multiple-well case

Multiple-well case (e.g. in the martensitic phase of shape memory alloys): Schmidt (2008)

$$\tilde{\Sigma}_\delta := \bigcup_{S \in \Sigma} SO(n)(I + \delta S) \quad \text{for some finite set of positive matrices } \Sigma \subset \mathbb{R}^{n \times n}_{\text{sym}}.$$

Suppose $W_\delta : \mathbb{R}^n \times \mathbb{R}^{n \times n} \to \mathbb{R}$ are

- Carathéodory, frame indifferent,
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$$V_\delta : \Omega \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}, \quad V_\delta(x,Y) := \frac{1}{\delta^2} W_\delta(x, I+\delta Y).$$

If $V_\delta \to V$ uniformly in x and locally uniformly Y, and $V(x,Y) \leq \mathfrak{U}(|Y|^2+1)$, then

$$\frac{1}{\delta^2}\int_\Omega W_\delta\Big(x, I+\delta\nabla u(x)\Big)\,dx \quad \xrightarrow{\Gamma} \quad \int_\Omega V^{\mathrm{qcls}}\Big(x, \mathfrak{E}u(x)\Big)\,dx$$

with

$$V^{\mathrm{qcls}}(x,Y) = \inf_{\varphi\in C_c^\infty(\mathbb{I}^n;\mathbb{R}^n)} \int_{\mathbb{I}^n} V\Big(x, Y+\mathfrak{E}\varphi(y)\Big)\,dy.$$

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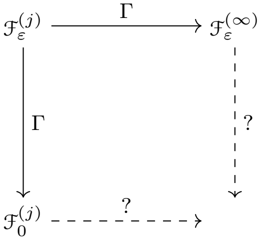
Our starting point: In case we have a fine periodic structure, do we also get a commuting diagram?

Commutability of Γ-limits

$$\begin{array}{ccc} \mathcal{F}_\varepsilon^{(j)} & \xrightarrow{\Gamma} & \mathcal{F}_\varepsilon^{(\infty)} \\ \Big\downarrow{\scriptstyle \Gamma} & & \\ \mathcal{F}_0^{(j)} & & \end{array}$$

We have some doubly indexed family of functionals. For one index fixed they Γ-converge.



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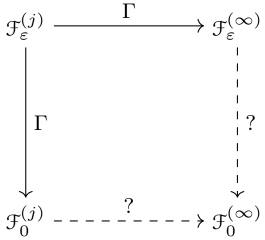

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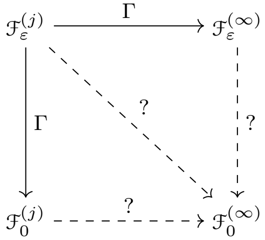

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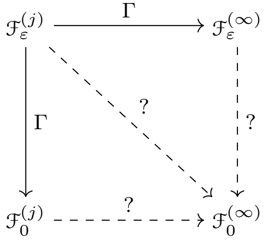
Commutability of Γ-limits and Γ-closure



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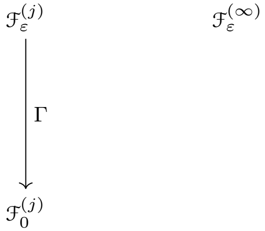

We have some sequence of Γ-converging families of functionals and another family.

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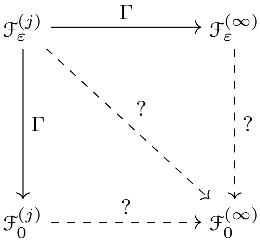

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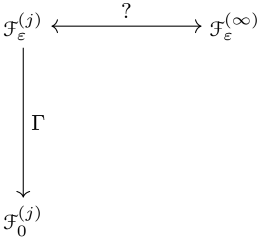

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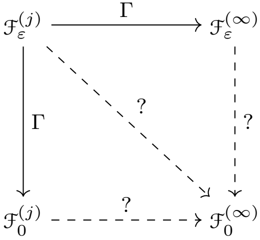

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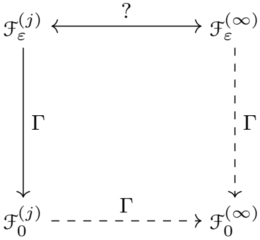

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Basic framework

Our functionals will be of form

$$\mathcal{F}(u) := \begin{cases} \int_\Omega f(x, \nabla u(x))\, dx & u \in W^{1,p}(\Omega; \mathbb{R}^m) \\ \infty & u \in L^p(\Omega; \mathbb{R}^m) \setminus W^{1,p}(\Omega; \mathbb{R}^m) \end{cases}$$

with Ω open bounded in $\mathbb{R}^n$ and $1 < p < \infty$.

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Definition

- A family $f_\varepsilon^{(j)} : \Omega \times \mathbb{R}^{m \times n} \to \mathbb{R}$ fulfils a standard p-growth condition if there are $\alpha, \beta > 0$ independent of j and ε such that

$$\alpha |X|^p - \beta \leq f_\varepsilon^{(j)}(x, X) \leq \beta(|X|^p + 1)$$

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- Families

$$\{\{f_\varepsilon^{(j)}\}_{\varepsilon>0}\}_{j\in\mathbb{N}} \quad \text{and} \quad \{f_\varepsilon^{(\infty)}\}_{\varepsilon>0}$$

  are equivalent on $U \subset \Omega$ open, if

$$\lim_{j\to\infty} \limsup_{\varepsilon\to 0} \int_U \sup_{|X|\leq R} |f_\varepsilon^{(j)}(x, X) - f_\varepsilon^{(\infty)}(x, X)|\, dx = 0$$

  for every $R \geq 0$.

Basic result on Γ-closure

Theorem (Γ-closure on a single domain)

Suppose that the family of Borel functions

$$f_\varepsilon^{(j)} : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R},\ j \in \mathbb{N} \cup \{\infty\},\ \varepsilon > 0,$$

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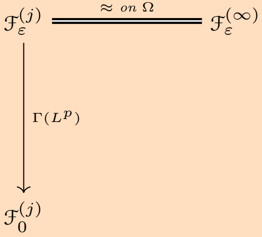

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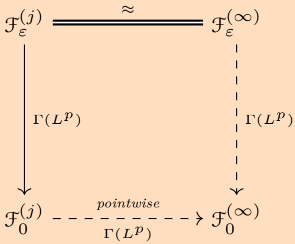

Tools for the proof

Key tool: Kristensen (1994); Fonseca, Müller, Pedregal (1998)

Lemma (Decomposition lemma)

Let $\{u_i\}_{i\in\mathbb{N}}$ be a bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$. There exists a subsequence $\{u_{i_k}\}_{k\in\mathbb{N}}$ and a sequence $\{v_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$\lim_{k\to\infty} |\{\nabla v_k \neq \nabla u_{i_k}\} \cup \{v_k \neq u_{i_k}\}| = 0$$

and $\{|\nabla v_k|^p\}_{k\in\mathbb{N}}$ is equi-integrable.

Moreover, if $u_i \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$, then the v_k can be chosen in such a way that $v_k = u$ on $\partial\Omega$ and $v_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$.

Idea for the proof

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\begin{array}{ccc}
\mathcal{F}_{\varepsilon}^{(j)} & \overset{\approx}{=\!=\!=} & \mathcal{F}_{\varepsilon}^{(\infty)} \\
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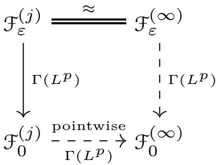

- Assume the vertical Γ-convergence.
- For every $u \in L^p(\Omega; \mathbb{R}^m)$

$$\limsup_{j \to \infty} \mathcal{F}_0^{(j)}(u) \leq \mathcal{F}_0^{(\infty)}(u).$$

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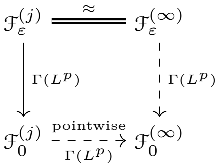

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- Justification of our assumption by Urysohn property and pointwise convergence below.

Possible extensions

- variable domain ✓

  If we assume equivalence and Γ-convergence on every open subset, then we get more information on density.

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- boundary values ✓
- upper bound ✓

  It may be relaxed for $j < \infty$.

Gårding type functionals

Definition

We say that the family of integral functionals

$$\mathcal{F}_\varepsilon^{(j)}: \ j \in \mathbb{N} \cup \{\infty\}, \ \varepsilon > 0$$

with densities $f_\varepsilon^{(j)}$ is of uniform p-Gårding type on $U \subset \Omega$, if

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- there are $\alpha_U > 0$, $\gamma_U \in \mathbb{R}$ such that

$$\mathcal{F}_\varepsilon^{(j)}(u, U) \geq \alpha_U \int_U |\nabla u(x)|^p \, dx - \gamma_U \int_U |u(x)|^p \; dx$$

  for all $u \in W^{1,p}(U; \mathbb{R}^m)$.

Γ-closure for Gårding type functionals

Theorem

Suppose that the family of Borel functions

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is of uniform p-Gårding type.

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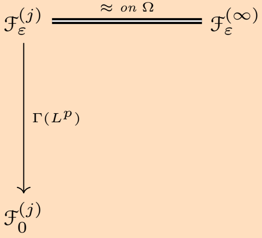

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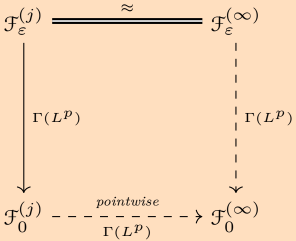


Addition trick

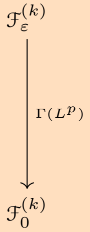
Proposition

Suppose that the family $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ with densities f_ε is of uniform p-Gårding type on Ω. Define for some null sequence $\lambda_k \searrow 0$

$$f_\varepsilon^{(k)}(x,X) := f_\varepsilon(x,X) + \lambda_k |X|^p$$

and denote by $\mathcal{F}_\varepsilon^{(k)}$ the corresponding integral functionals.

If



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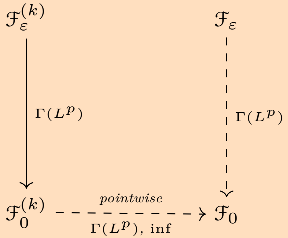
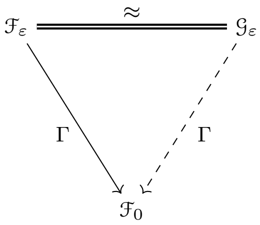


Consequences

- Perturbation:



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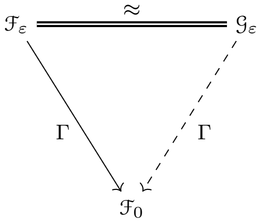

Homogenization closure by Braides (1986)

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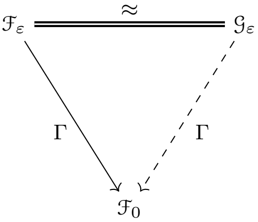

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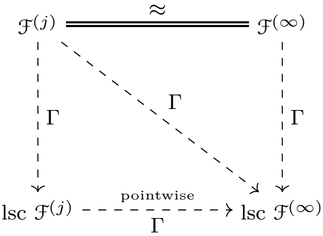

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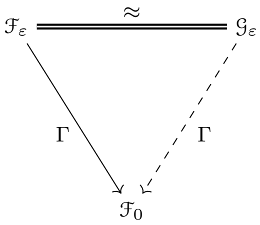

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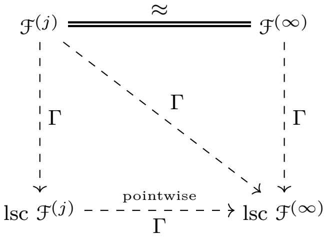

linearization in the one-well case by Dal Maso, Negri, Percivale
geometric linearization in the multiple-well case by Schmidt

Γ-commuting diagrams

Theorem (Commutability)

Suppose that the family of functionals

$$\mathcal{F}_\varepsilon^{(j)},\ j \in \mathbb{N} \cup \{\infty\},\ \varepsilon > 0,$$

with densities $f_\varepsilon^{(j)}$ is of uniform p-Gårding type on Ω.

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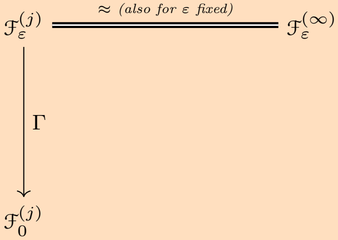

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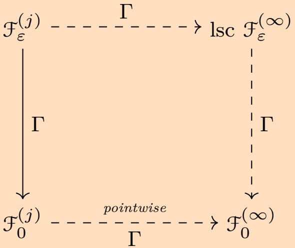

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- Müller, Neukamm
- our setting

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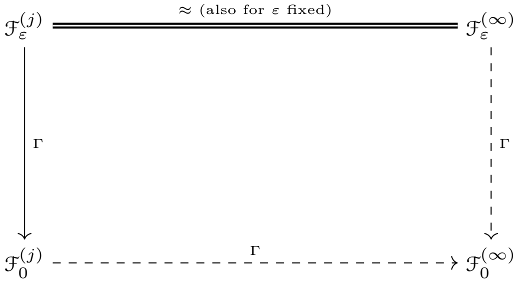
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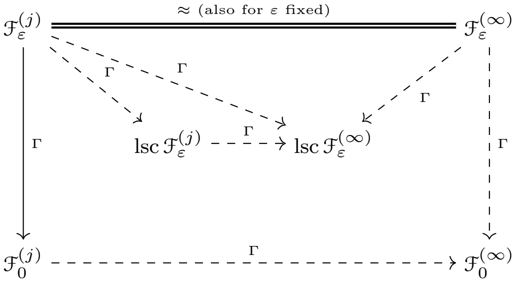

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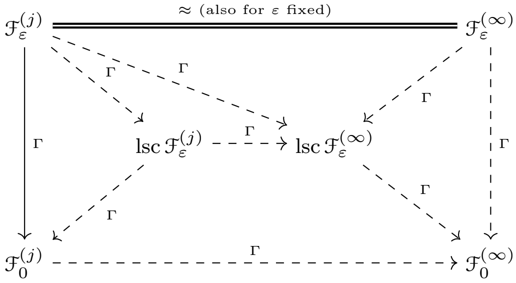

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Simultaneous limits

Theorem

Let $\Omega \subset \mathbb{R}^n$ be bounded and open. Suppose that the family of functionals $\mathcal{F}_\varepsilon^{(j)}$, $j \in \mathbb{N} \cup \{\infty\}$, $\varepsilon > 0$, with densities $f_\varepsilon^{(j)} : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is of uniform p-Gårding type on Ω. Let us have $\{j_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ and $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k \searrow 0$. Assume that

- $\Gamma(L^p)\text{-}\lim\limits_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{(\infty)} = \mathcal{F}_0^{(\infty)}$,
- $\lim\limits_{k\to\infty} \int_\Omega \sup\limits_{|X|\le R} |f_{\varepsilon_k}^{(j_k)}(x,X) - f_{\varepsilon_k}^{(\infty)}(x,X)|\, dx = 0$ *for every* $R > 0$.

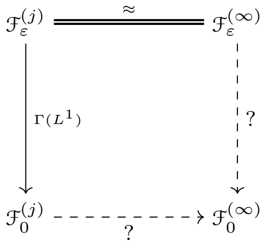
Then

$$\Gamma(L^p)\text{-}\lim_{k\to\infty} \mathcal{F}_{\varepsilon_k}^{(j_k)} = \mathcal{F}_0^{(\infty)}.$$

E.g., this condition is surely satisfied if

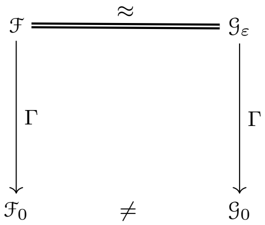
$$\forall R > 0 : \lim_{j\to\infty} \sup_{\varepsilon>0} \int_\Omega \sup_{|X|\le R} |f_\varepsilon^{(j)}(x,X) - f_\varepsilon^{(\infty)}(x,X)|\, dx = 0$$

$p = 1$ case



Now suppose $p = 1$. Then Γ-limits below would be finite on $BV(\Omega; \mathbb{R}^m)$.

The counterexample is based on Bouchitte, Dal Maso (1993). It shows that the consequence regarding perturbation cannot hold:



Counterexample for $p = 1$ case

Let us consider the scalar case $m = 1$ with $\Omega := I = (-1, 1)$. Take

$$f : \mathbb{R} \to \mathbb{R}, \quad f(\xi) := \max\{|\xi|, 2|\xi| - 1\}.$$

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- convex,
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The constant sequence of functionals on $L^1(I)$ given by $\mathcal{F}$,

$$\mathcal{F}(u) := \begin{cases} \int_{-1}^{1} f(u'(x))\, dx, & u \in W^{1,1}(I), \\ \infty, & \text{else on } L^1(I), \end{cases}$$

Γ-converges to its relaxation $\mathcal{F}_0$ that is finite exactly on $BV(I)$ and there takes the values

$$\begin{aligned} \mathcal{F}_0(u) &= \int_{-1}^{1} f(u'(x))\, dx + \int_{-1}^{1} \underline{f}\left(\frac{dD^s u}{d|D^s u|}(x)\right) d|D^s u|(x) \\ &= \int_{-1}^{1} f(u'(x))\, dx + 2|D^s u|(I), \end{aligned}$$

Counterexample for $p = 1$ case (2)

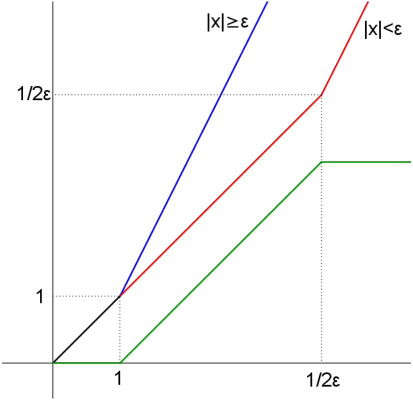
Take

$$g_\varepsilon(x,\xi) := f\left(\frac{\xi}{a_\varepsilon(x)}\right) a_\varepsilon(x)$$

where

$$a_\varepsilon(x) := \begin{cases} 1, & |x| \geq \varepsilon, \\ \frac{1}{2\varepsilon}, & |x| < \varepsilon. \end{cases}$$

The (one-index) family $\{g_\varepsilon\}_{\varepsilon>0}$ is equivalent to the constant family given by f:



$$\begin{aligned} \limsup_{\varepsilon\to 0}\int_{-1}^{1}\sup_{|\xi|\leq R}|g_\varepsilon(x,\xi)-f(\xi)|\,dx &= \limsup_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\sup_{|\xi|\leq R}\left|\tfrac{1}{2\varepsilon}f(2\varepsilon\xi)-f(\xi)\right|\,dx \\ &\leq \limsup_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\sup_{|\xi|\leq R}4|\xi|\,dx \\ &= \limsup_{\varepsilon\to 0} 8R\varepsilon \\ &= 0 \end{aligned}$$

Counterexample for $p = 1$ case (3)

Let us denote $\lambda_\varepsilon := a_\varepsilon \mathcal{L}^1 \in M(I)$. Since for $u \in W^{1,1}(I)$

$$Du = u' \mathcal{L}^1 = \frac{u'}{a_\varepsilon} \lambda_\varepsilon$$

the corresponding functionals $\mathcal{G}_\varepsilon$ have on $W^{1,1}(I)$ the following representation

$$\mathcal{G}_\varepsilon(u) = \int_{-1}^{1} g_\varepsilon(x, u'(x))\, dx = \int_{-1}^{1} f\left(\tfrac{dDu}{d\lambda_\varepsilon}(x)\right) d\lambda_\varepsilon(x).$$

Since

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$$\lambda_\varepsilon \overset{*}{\rightharpoonup} \lambda := \delta_0 + \mathcal{L}^1 \quad \text{in } M(I)$$

it follows by the results from Buttazzo, Freddi (1991) that $\mathcal{G}_\varepsilon$ $\Gamma(L^1)$-converges to $\mathcal{G}_0$ given by

$$\mathcal{G}_0(u) = \int_{-1}^{1} f\left(\frac{dD^a_\lambda u}{d\lambda}(x)\right) d\lambda(x) + \int_{-1}^{1} \underline{f}\left(\frac{dD^s_\lambda u}{d|D^s_\lambda u|}(x)\right) d|D^s_\lambda u|(x)$$

if $u \in BV(I)$ and ∞ otherwise, where

$$Du = D^a_\lambda u + D^s_\lambda u, \quad D^a_\lambda u \ll \lambda, \quad D^a_\lambda u \perp \lambda.$$

Counterexample for $p = 1$ case (4)

Therefore, for $u \in BV(I)$

$$
\begin{aligned}
\mathcal{F}_0(u) &= \int_{-1}^{1} f(u'(x))\, dx + 2\|D^s u\|, \\
\mathcal{G}_0(u) &= \int_{-1}^{1} f\left(\frac{dD^a_\lambda u}{d\lambda}(x)\right) d\lambda(x) + 2\|D^s_\lambda u\|.
\end{aligned}
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$$\begin{aligned} \mathcal{F}_0(u) &= \int_{-1}^{1} f(u'(x))\, dx + 2\|D^s u\|, \\ \mathcal{G}_0(u) &= \int_{-1}^{1} f\left(\frac{dD^a_\lambda u}{d\lambda}(x)\right) d\lambda(x) + 2\|D^s_\lambda u\|. \end{aligned}$$

Choose $u := \chi_{(0,1)} \in SBV(I)$. Then $Du = \delta_0$ and

$$\begin{aligned} D^a u + D^s u &= 0 \cdot \mathcal{L}^1 + \delta_0, \\ D^a_\lambda u + D^s_\lambda u &= \chi_{\{0\}} \cdot \lambda + 0. \end{aligned}$$

Hence

$$\mathcal{F}_0(\chi_{(0,1)}) = 2 \quad \text{and} \quad \mathcal{G}_0(\chi_{(0,1)}) = 1.$$

Additional condition at ∞

Problems: families whose difference grows linearly for large X at least on some set of x.

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Definition

Families $\{\{f_\varepsilon^{(j)}\}_{\varepsilon>0}\}_{j\in\mathbb{N}}$ and $\{f_\varepsilon^{(\infty)}\}_{\varepsilon>0}$ are equivalent at ∞ on $U \subset \Omega$, if for

$$r_\varepsilon^{(j)}(R) := \operatorname*{ess\,sup}_{x\in U} \sup_{|X|\geq R} \frac{|f_\varepsilon^{(j)}(x,X) - f_\varepsilon^{(\infty)}(x,X)|}{|X|}$$

it holds

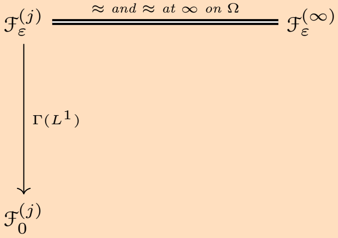
$$\lim_{R\to\infty} \limsup_{j\to\infty} \limsup_{\varepsilon\to 0} r_\varepsilon^{(j)}(R) = 0.$$

Corresponding basic result for $p = 1$

Theorem (Γ-closure on a single domain)

Let $\Omega \subset \mathbb{R}^n$ be bounded and open. Suppose that the family of Borel functions $f_\varepsilon^{(j)} : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$, $j \in \mathbb{N}$, $\varepsilon > 0$, uniformly fulfils a standard linear growth condition. Assume that

If



then