Infinite pinning of interfaces in a random elastic medium

Martin Jesenko

joint work with Patrick Dondl

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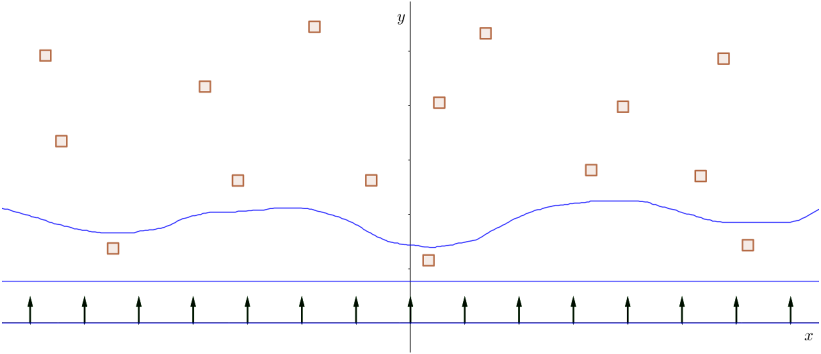
Moving of interfaces



An interface moves through a media driven by a constant force F upwards. In the media, there are obstacles that act with a force $f(x, y)$ downwards.

We suppose the interface to move accroding to the curvature flow. Thus

$$v_n = \varkappa - f + F.$$

Starting with a flat interface $u = 0$ at time 0, we suppose the interface to be given by a graph at all times. Assuming the gradient to be small, we arrive at

$$\frac{\partial u}{\partial t}(x, t) = \triangle u(x, t) - f(x, u(x, t)) + F$$

Deterministic periodic setting

Theorem (Dirr, Yip)

If the array of obstacles is deterministic and 1-perodic (positions and strengths),

Dirr, N.; Yip, N. K. Pinning and de-pinning phenomena in front propagation in heterogeneous media. Interfaces Free Bound. 8 (2006), no. 1, 79–109.

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If the array of obstacles is deterministic and 1-perodic (positions and strengths), then there exists $F_* > 0$ such that

Pinning

for any $0 \leq F \leq F_*$ there exists a stationary solution $U_F > 0$, and because of the comparison principle the interface "gets pinned" under the graph of U_F.

Depinning

for any $F > F_*$ there exists an unique $T_F > 0$ and a spatially 1-periodic solution U_F with

$$U_F(\_, t + T_F) = U_F(\_, t) + 1.$$

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Quenched Edwards-Wilkinson model

We suppose that the obstacles have the same shape and (time-independent) random positions and strenghts. More precisely, the moving is determined by the equation

$$\frac{\partial u(x,t,\omega)}{\partial t} = \triangle u(x,t,\omega) - f(x, u(x,t,\omega),\omega) + F$$

with the force of the obstacle field being the random function

$$f(x,y,\omega) = \sum_i f_i(\omega)\varphi(x - x_i(\omega), y - y_i(\omega)).$$

We assume

Condition 1 (Shape of obstacles)

There exist $r_0, r_1 > 0$ with $r_1 > \sqrt{n}r_0$ so that

$$\varphi(x,y) \geq 1 \text{ for } \|(x,y)\|_\infty \leq r_0 \quad \text{and} \quad \varphi(x,y) = 0 \text{ for } \|(x,y)\| \geq r_1.$$

Condition 2 (Obstacle positions and strengths)

The random distribution of obstacle sites $\{(x_i, y_i)\}_{i\in\mathbb{N}} \subset \mathbb{R}^n \times [r_1, \infty)$ and strengths $\{f_i\}_{i\in\mathbb{N}} \subset [0,\infty)$ satisfy:

- $\{(x_i, y_i)\}_{i\in\mathbb{N}}$ are distributed according to an $(n+1)$-dimensional Poisson point process on $\mathbb{R}^n \times [r_1, \infty)$ with intensity $\lambda > 0$,
- $\{f_i\}_{i\in\mathbb{N}}$ are independent and identically distributed strictly positive random variables that are independent of $\{(x_i, y_i)\}_{i\in\mathbb{N}}$.

Pinning result

Theorem (Dirr, Dondl, Scheutzow)

If Conditions 2.1 and 2.2 are satisfied, then there exists $F_* > 0$ and a non-negative $v : \mathbb{R}^n \times \Omega \to [0, \infty)$ so that

$$0 \geq \triangle v(x, \omega) - f(x, v(x, \omega), \omega) + F_*$$

almost surely. Hence, for $F \leq F_*$ any solution for Quenched Edwards-Wilkinson model with trivial initial condition gets pinned.

Dirr, N.; Dondl, P. W.; Scheutzow, M. Pinning of interfaces in random media. Interfaces Free Bound. 13 (2011), no. 3, 411–421

Dondl, P. W.; Scheutzow, M. Positive speed of propagation in a semilinear parabolic interface model with unbounded random coefficients. Netw. Heterog. Media 7 (2012), no. 1, 137–150

Infinite pinning

Is there a reasonable distribution of obstacle strengths in order for infinite pinning to occur? That is, can it happen that for **any** $F > 0$ a supersolution to the corresponding equation exists?

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Theorem

Let us suppose that there exist $\alpha, \varepsilon > 0$ such that for all M sufficiently large

$$\mathbf{P}(f_1 \geq M) \geq \frac{\alpha M^{\varepsilon}}{M^{\frac{1}{2}+\frac{1}{n}}}.$$

Then the infinite pinning occurs almost surely.

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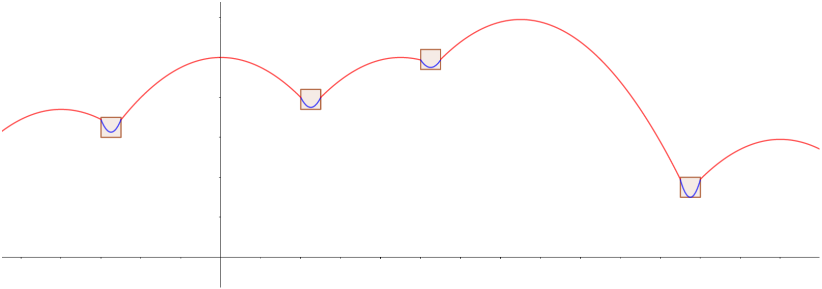
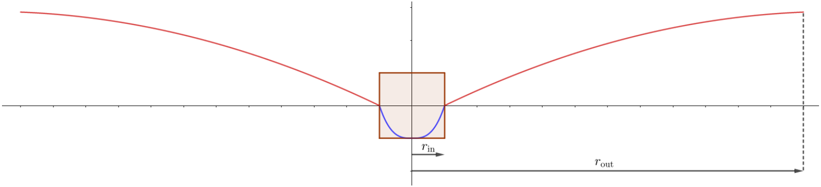
A supersolution v has to fulfil

$$\triangle v \leq f(x,v) - F \quad \text{and} \quad v > 0.$$



Local solution



We construct a radial symmetric function from two parts that meet at $(r_{in}, 0)$. Let $m \in \mathbb{N} \cup \{0\}$, $r_{in} := r_0$ and $q := r_{out}/r_{in}$.

v_{in}

$$\triangle v_{in} = F_{in} \left(\frac{r}{r_{in}}\right)^m$$

Therefore,

$$v_{in}(r) = \frac{F_{in}(r^{m+2} - r_{in}^{m+2})}{(m+n)(m+2)r_{in}^m}.$$

Then

$$v_{in}(0) = \frac{-F_{in} r_{in}^2}{(m+n)(m+2)},$$

$$v_{in}'(r_{in}) = \frac{F_{in} r_{in}}{m+n}.$$

v_{out}

$$\triangle v_{out} = -F_{out}, \quad v_{out}'(r_{in}) = 0.$$

The derivative is

$$v_{out}'(r) = \frac{F_{out}}{nr^{n-1}}(r_{out}^n - r^n).$$

At the inner boundary, we have

$$v_{out}'(r_{in}) = \frac{F_{out}}{nr_{in}^{n-1}}(r_{out}^n - r_{in}^n)$$

$$= \frac{F_{out} r_{in}}{n}(q^n - 1).$$

Local solution

Proposition

The function

$$v_{\text{local}} : \mathbb{R}^n \to \mathbb{R}, \quad v_{\text{local}}(x) := \begin{cases} v_{\text{in}}(|x|), & |x| < r_{\text{in}} \\ v_{\text{out}}(|x|), & r_{\text{in}} \le |x| < r_{\text{out}} \\ \lim_{r \to r_{\text{out}}} v_{\text{out}}(r), & |x| = r_{\text{out}} \\ \infty, & |x| > r_{\text{out}} \end{cases},$$

- *is radially strictly increasing on* $B_{r_{\text{out}}}$,
- *and the graph of* $v_{\text{local}}|_{B_{r_{\text{in}}}}$ *is contained in* $B_{r_{\text{in}}} \times \left[\frac{-F_{\text{in}} r_{\text{in}}^2}{(m+n)(m+2)}, 0\right]$.

Suppose

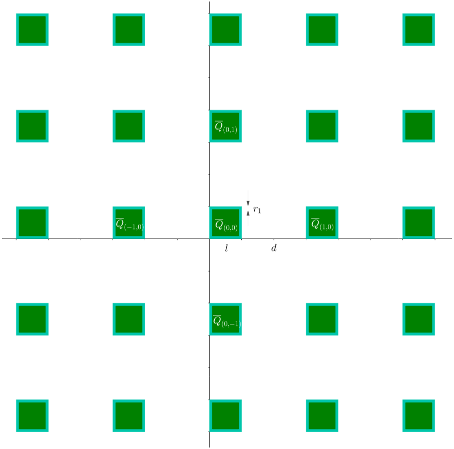
$$\frac{(m+n)(m+2)}{r_0} \ge F_{\text{in}} \ge F_{\text{out}} \frac{m+n}{n}(q^n - 1).$$

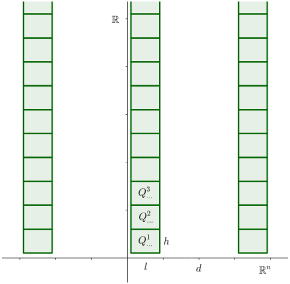
Then for any $M > 0$, v_{local} *suffices*

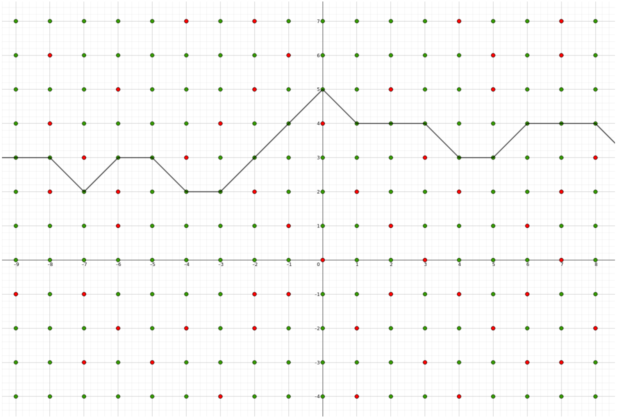
$$0 \ge \triangle v_{\text{local}} - M\varphi(\_, v_{\text{local}}(\_)) + F \quad \text{on } B_{r_{\text{out}}}.$$

for all $F \le \min\{F_{\text{out}}, M - F_{\text{in}}\}$.

Partition of the space







Theorem (Dirr, Dondl, Grimmett, Holroyd, Scheutzow)

Suppose $z \in \mathbb{Z}^{n+1}$ is open with probability $p \in (0,1)$ and closed otherwise, with different sites receiving independent states. If $p > 1 - \frac{1}{(2n+2)^2} =: p_c$, then there exists a.s. a (random) function $L : \mathbb{Z}^n \to \mathbb{N}$ with the following properties:

- *For each $a \in \mathbb{Z}^n$, the site $(a, L(a)) \in \mathbb{Z}^{n+1}$ is open.*
- *For any $a, b \in \mathbb{Z}^n$ with $\|a - b\|_1 = 1$, we have $|L(a) - L(b)| \leq 1$.*

Percolation result applied

Fix $h > 0$ and $M \gg 1$. Probability that in a cuboid with volume $(l - 2r_1)^n h$ there is a centre of an obstacle with strength at least M is $1 - e^{-\lambda(l-2r_1)^n h \mathbf{P}(f_1 \geq M)}$. Thus, if

$$1 - e^{-\lambda(l-2r_1)^n h \mathbf{P}(f_1 \geq M)} > 1 - \frac{1}{(2n+2)^2}, \quad \text{i.e.} \quad l > 2r_1 + \left(\frac{2\log(2n+2)}{\lambda h \mathbf{P}(f_1 \geq M)}\right)^{1/n},$$

then the Percolation theorem is applicable. Therefore, let us take

$$l(h, M) := 2r_1 + C_0 \frac{M^{(\frac{1}{2}+\frac{1}{n}-\varepsilon)\cdot\frac{1}{n}}}{h^{\frac{1}{n}}} \quad \text{with} \quad C_0 := \left(\frac{3\log(2n+2)}{\lambda\alpha}\right)^{1/n}.$$

According to the Percolation theorem, there exists a.s. a random function $L : \mathbb{Z}^n \times \Omega \to \mathbb{N}$ such that

- for each $a \in \mathbb{Z}^n$, the cuboid $Q_a^{L(a)}$ contains a center of an obstacle (x_a, y_a) with strength at least M,
- for any $a, b \in \mathbb{Z}^n$ with $\|a - b\|_1 = 1$, we have $|L(a) - L(b)| \leq 1$.

Flat supersolution

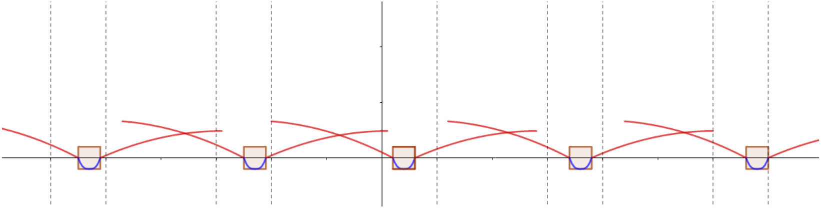
Fix now also d, and let $r_{\text{out}} := \sqrt{n}\left(l + \frac{d}{2} - r_1\right)$. We define the flat supersolution

$$v_{\text{flat}} : \mathbb{R}^n \times \Omega \to \mathbb{R}, \quad v_{\text{flat}}(x,\omega) := \min_{a \in \mathbb{Z}^n} v_{\text{local}}(x - x_a(\omega)).$$

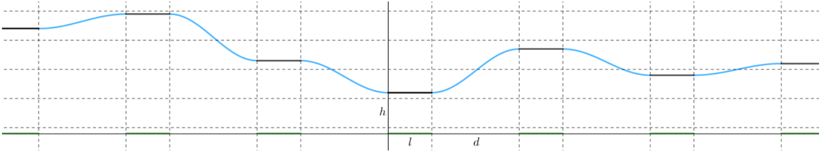
If F_{in} and F_{out} satisfy the $\square$-inequality, then v_{flat} satisfies a.s. in the sense of distributions (and in the sense of viscosity solutions)

$$0 \geq \triangle v_{\text{flat}}(x) - \sum_{a \in \mathbb{Z}^n} M\varphi(x - x_a, v_{\text{flat}}(x)) + F$$

for all $F \leq \min\{F_{\text{out}}, M - F_{\text{in}}\}$.



Gluing function



Proposition

Let $h, l, d > 0$. Suppose $y : \mathbb{Z}^n \to \mathbb{R}$ has the following property:

$$\forall a, b \in \mathbb{Z}^n : \|a - b\|_1 = 1 \;\Rightarrow\; |y_a - y_b| < 2h.$$

Then there exists $C = C(n) > 0$ and a smooth function $v_{\mathrm{glue}} : \mathbb{R}^n \to \mathbb{R}$ such that

- $v_{\mathrm{glue}}|_{\overline{Q}_a} = y_a$ *for every* $a \in \mathbb{Z}^n$,
- $\operatorname{supp} \nabla v_{\mathrm{glue}} \subset \overline{D}$,
- $\|D^2 v_{\mathrm{glue}}\|_\infty \le C \frac{h}{d^2}$,
- $\|\nabla v_{\mathrm{glue}}\|_\infty \le C \frac{h}{d}$.

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(local solutions for different obstacles meet).

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  (local solutions for different obstacles meet).
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We still need to fulfil the following inequality:

$$\frac{F_{\text{in}} n}{m+n} \geq |F_{\text{out}}|(-1+q^n) \quad \text{or} \quad m+2 \geq 2C\frac{r_0}{n}\frac{h}{d^2}(-1+q^n)$$

(the derivative falls at the point of non-differentiability)

Proof of theorem (cont.)

Since

$$2C\frac{r_0}{n}q^n = 2C\frac{r_{\text{out}}^n}{nr_0^{n-1}} \leq C'\left(\frac{m^{1+\frac{2}{n}-2\varepsilon}}{h} + \left(\frac{d}{2}+r_1\right)^n\right)$$

with $C' = C'(n, \lambda, \alpha, r_0)$, the desired inequality will hold if the following is fulfilled:

$$m \geq C'\frac{h}{d^2}\left(\frac{m^{1+\frac{2}{n}-2\varepsilon}}{h} + \left(\frac{d}{2}+r_1\right)^n\right) = C'\left(\frac{m^{1+\frac{2}{n}-2\varepsilon}}{d^2} + \frac{h}{d^2}\left(\frac{d}{2}+r_1\right)^n\right)$$

This may be achieved by taking

$$d^2 := 2C'm^{\frac{2}{n}-2\varepsilon} \quad \text{and} \quad h := \frac{1}{2(2C')^{\frac{n}{2}}}m^{\frac{2}{n}+(n-2)\varepsilon}.$$

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Then

$$F_{\text{out}} = 2C\frac{h}{d^2} = \frac{C}{2(2C')^{\frac{n}{2}}}m^{n\varepsilon},$$

and for any $0 < F \leq \frac{1}{2}\min\{F_{\text{out}}, M\}$, the function $v := v_{\text{flat}} + v_{\text{glue}}$ satisfies a.s.

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$$0 \ge \triangle v(x,\omega) - f(x, v(x,\omega),\omega) + F.$$

To conclude: Since for any $F > 0$ there is $m \in \mathbb{N}$ such that

$$F \le \min\left\{\frac{C}{4(2C')^{\frac{n}{2}}}m^{n\varepsilon}, \frac{(m+n)(m+2)}{r_0}\right\},$$

the construction above yields an appropriate supersolution. Hence, the infinte pinning takes place.