Existence of a blocking line in the quenched Edwards-Wilkinson model

Martin Jesenko

joint work with
Patrick Dondl (Freiburg) & Michael Scheutzow (TU Berlin)

Calculus of Variations on Schiermonnikoog
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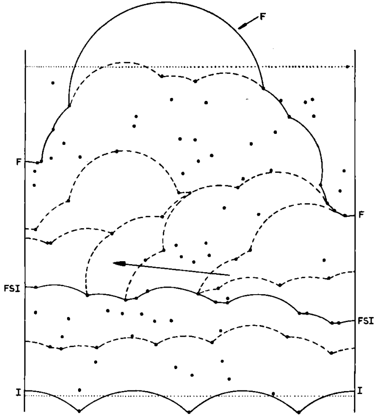
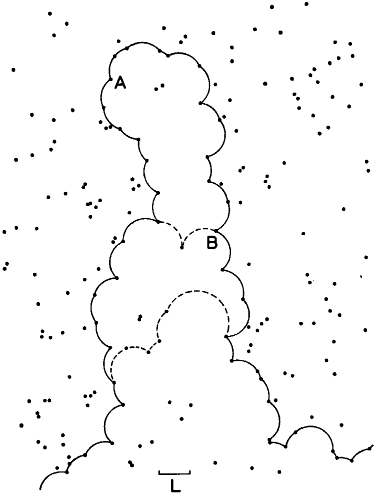
History

Foreman, Makin. Dislocation movement through random arrays of obstacles, Philosophical Magazine, 14:131 (1966)





Background

Brazovskii, Nattermann. Pinning and sliding of driven elastic systems: from domain walls to charge density waves, Advances in Physics 53 (2004), no. 2

- Domain walls in magnets
- Isolated magnetic flux lines or dislocation lines
- CDWs (Charge density waves)

Energy:

$$E(u) = \int \left(C|\nabla u(x)|^2 + V(x, u(x)) - Fu(x) \right) \, dx$$

Equation:

$$\partial_t u(t,x) = \Delta u(t,x) - f(x, u(t,x)) + F, \quad u(0, \_) = 0$$

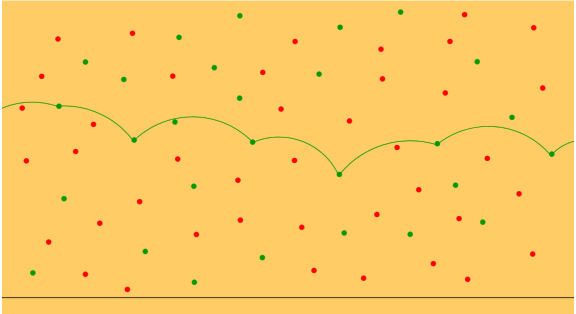
with f being a random force (with zero mean) and F an external deterministic force.

Dirr, Yip. Pinning and de-pinning phenomena in front propagation in heterogeneous media. Interfaces Free Bound. 8 (2006)

Dirr, Dondl, Scheutzow. Pinning of interfaces in random media. Interfaces Free Bound. 13 (2011)

Dondl, Scheutzow, Throm. Pinning of interfaces in a random elastic medium and logarithmic lattice embeddings in percolation. Proc. Roy. Soc. Edinburgh Sect. A 145 (2015)

Model



In this case, we have an array of positive and negative obstacles that now have positive diameter and random positions. The positive ones are strong enough to produce an inclination k (at the exit), which we fix. We will show that for sufficiently small F there exists a.s. a viscosity supersolution of

$$v''(x) - f(x, v(x)) + F \leq 0.$$

The function v will be piecewise quadratic, and in points of non-differentiability the condition on viscosity solution will be trivially met as it will hold

$$\lim_{x \nearrow a} v'(x) \geq \lim_{x \searrow a} v'(x).$$

The diameters of obstacles are variable; our goal is actually to determine them appropriately so that the pinning takes place. Let us clearly state our assumptions:

- The centers of the positive/negative obstacles $(x_j^{\pm}, y_j^{\pm})$ are distributed according to the Poisson point process with parameter $\lambda^{\pm}$.
- All positive obstacles have full strength (at least) S in the squares of side 2ρ centered at (x_j^+, y_j^+), to which we later refer to as *cores*.
- The impact of every obstacle vanishes completely outside a ball $B_{\alpha\rho}(x_j^{\pm}, y_j^{\pm})$ with $\alpha > \sqrt{2}$ being fixed.

We will in fact construct a function v with

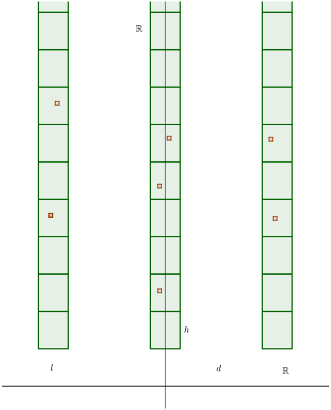
$$v''(x) - \tilde{f}(x, v(x)) + F \leq 0$$

where

$$\tilde{f}(x,y) = \begin{cases} \displaystyle\sum_{j \in \mathbb{N}} S\chi_{[-1,1]^2}\left(\frac{x - x_j^+}{\rho}, \frac{y - y_j^+}{\rho}\right), & \text{if } \forall i \in \mathbb{N} : \operatorname{dist}((x,y),(x_i^-, y_i^-)) > \alpha\rho \\ -\infty & \text{else} \end{cases}$$

Namely, our supersolution will completely avoid negative obstacle, and therefore, their precise strength does not matter. As for the positive obstacles, we will only employ that they exert force (at least) S in the cores.

Localization



For every $i \in \mathbb{Z},\ j \in \mathbb{N}$ let

$$Q_{i,j} := \Big([-\tfrac{l}{2}, \tfrac{l}{2}] + i(l+d)\Big) \times [jh, (j+1)h].$$

We start at height h so that there is no intersection of localized positive obstacles with the x-axis. (The condition reads $h > \rho(\alpha - 1)$.) The condition is that the center lies in

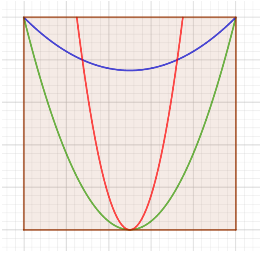
$$\tilde{Q}_{i,j} := \Big([-\tfrac{l}{2}+\rho, \tfrac{l}{2}-\rho]+i(l+d)\Big)\times[jh, (j+1)h]$$

so that the cores do not intersect the free space.

Inside a core

Within the core of an obstacle, the condition reads $v''(x) \leq S - F$. Let us suppose $F \leq \frac{S}{2}$ (since in fact, as we will see, we may only pin $F \ll S$). The maximal slope at the exit of a parabola $v'' = \frac{S}{2}$ having the vertex on the mid vertical line of the core is

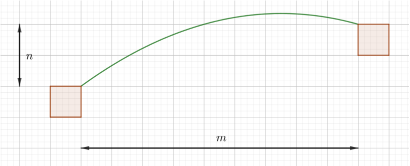
$$\min\{\tfrac{S}{2}\rho, \sqrt{2S\rho}\}.$$



The second expression follows from the fact that a large force results in exiting the obstacle at the upper edge and thus not exploiting it fully (red line). For $k \leq 4$, however, this does not happen, and we have $k = \frac{S}{2}\rho$. We will suppose even $k \leq 1$ anyway and take the parabola that exits the core at its upper corners. The cases $k = 1, 4$ are depicted in blue and green.

An upper bound on the pinned force

Let us first determine the force F that we may block when we have positive obstacles that allow for exiting inclinations up to k.



Let F be a given force. There exists a connecting function u with

$$u''(x) + F \leq 0$$

if and only if

$$\boxed{F \leq 2\frac{km - n}{m^2}}$$

If in our decomposition, we thus connect two obstacles in boxes $Q_{i,j}$ and $Q_{i+1,j+e}$ with $e \in \{-1, 0, 1\}$, then $d < m < 2l + d$ and $|n| < 2h$. For a given k, we get a positive force if

$$\boxed{kd > 2h} \tag{1}$$

We then block at least

$$\boxed{F \leq 2\frac{kd - 2h}{(d + 2l)^2}} \tag{2}$$

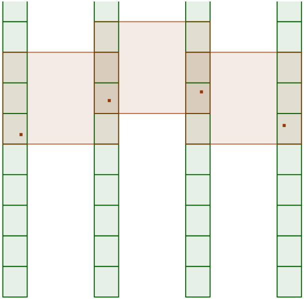
Position of vertex

The parabola

$$u(x) = \left(\frac{F}{2}m + \frac{n}{m}\right)x - \frac{F}{2}x^2$$

has its vertex in

$$x_0 = \frac{\frac{F}{2}m + \frac{n}{m}}{F}, \quad y_0 = \frac{\left(\frac{F}{2}m + \frac{n}{m}\right)^2}{2F}.$$



Let us set the admissible height of the parabola to the second rectangle above the lower obstacle. We want to achieve $y_0 \leq 2h$. Since

$$y_0 = \frac{F}{2}x_0^2 \leq \frac{F}{2}m^2,$$

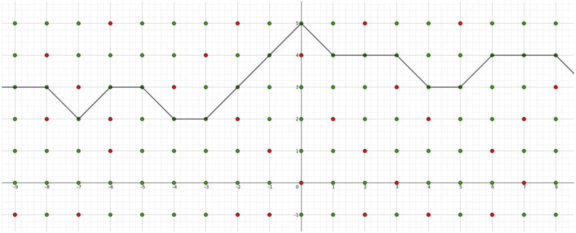
this will surely be fulfilled if

$$4h \geq Fm^2.$$

Therefore, a sufficient condition is

$$\boxed{F \leq \frac{4h}{(2l+d)^2}} \tag{3}$$

Lipschitz percolation



Theorem (Dirr, Dondl, Grimmett, Holroyd, Scheutzow)

Suppose $z \in \mathbb{Z}^{n+1}$ is open with probability $p \in (0,1)$ and closed otherwise, with different sites receiving independent states. If $p > 1 - \frac{1}{(2n+2)^2} =: p_c$, then there exists a.s. a (random) function $L : \mathbb{Z}^n \to \mathbb{N}$ with the following properties:

- *For each $a \in \mathbb{Z}^n$, the site $(a, L(a)) \in \mathbb{Z}^{n+1}$ is open.*
- *For any $a, b \in \mathbb{Z}^n$ with $\|a - b\|_1 = 1$, we have $|L(a) - L(b)| \leq 1$.*

Proposition (Scheutzow)

The result still hold (with some critical probability $p_D \in (0,1)$) if the states of sides may be dependent in the vertical direction, however, with a fixed finite range D.

Putting it together

Now let us state our problem in terms of the Lipschitz percolation. Our sites $(i,j) \in \mathbb{Z}^2$ are boxes $Q_{i,j}$. We impose three conditions on a box to be open:

- $\tilde{Q}_{i,j}$ must contain a center of a positive obstacle. Thus we get a condition on the heights and lengths. That means that $2\rho < l, h$ and that the probability is

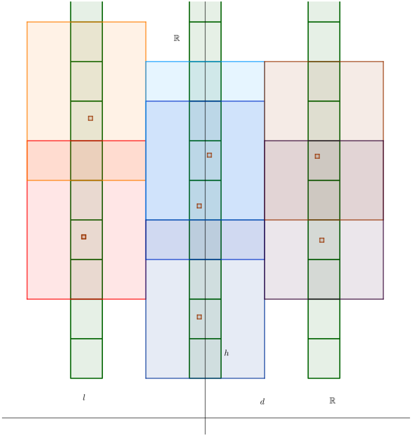
$$\mathbf{P}(\tilde{Q}_{i,j} \text{ contains a center of a positive obstacle}) = 1 - e^{-\lambda^{+}(l-2\rho)(h-2\rho)}.$$

- Moreover, a core itself together with a strip of width b around it must not intersect any negative obstacle. We need $b < d$ in order for probabilities to be independent in the horizontal direction and $b < 2h$ to have limited dependence (4 in-between for independence) in the vertical direction. The probability then reads

$$\mathbf{P}(\text{core with the strip intersects no negative obstacles}) = e^{-\lambda^{-}(2b+2\rho+2\alpha\rho)^2}.$$

- Lastly, we want the rectangle "around" a positive obstacle with length $l + d$ and height $4h$ (as on the picture) to contain less than N centers of negative obstacles. Here,

$$\mathbf{P}(\text{rectangle intersects less than } N \text{ negative obstacles}) \geq 1 - \frac{(4\lambda^{-}h(l+d))^N}{N!}.$$



Hence,

$$\mathbf{P}(Q_{i,j} \text{ is open}) \geq (1-e^{-\lambda^+(l-2\rho)(h-2\rho)})e^{-4\lambda^-(b+(1+\alpha)\rho)^2}\left(1-\frac{(4\lambda^- h(l+d))^N}{N!}\right).$$

We may surely employ the percolation result (with dependence) if the right-hand side is bigger than p_4.

The scales can be chosen in the following way:

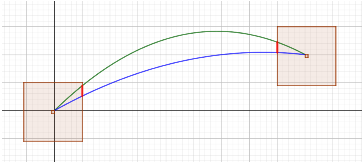
- Let us suppose $l, h \geq 4\rho$. Set $d := l$ and $h := \frac{kd}{4}$ (to obey (1)), and choose l so large that
$$1 - e^{-\lambda^+ (l-2\rho)(h-2\rho)} \geq 1 - e^{-\lambda^+ hl/4} \geq p_4^{1/3}.$$
- Suppose $(1+\alpha)\rho \leq b$. We choose b small enough so that $b < d$, $b < 2h$ and
$$e^{-4\lambda^- (2b)^2} \geq p_4^{1/3}.$$
- Finally, choose $N \in \mathbb{N}$ so that
$$1 - \frac{(4\lambda^- h(l+d))^N}{N!} > p_4^{1/3}.$$

The percolation result is now applicable, and we get a Lipschitz function between open sites. By (2) and (3) making the choices as above, we block at least

$$F^* := \min\left\{\frac{4h}{(2l+d)^2}, 2\frac{kd-2h}{(2l+d)^2}\right\} = \frac{k}{9l}.$$

However, our supersolution can still hit some negative obstacle (outside the b-strips)!

Excluding negative obstacles



If $u_1''(x) = -F_1$ and $u_2''(x) = -F_2$, then

$$u_1(\delta) - u_2(\delta) = \frac{F_1 - F_2}{2}\delta(m-\delta).$$

Since $k \leq 1$, the parabolas surely exit the squares at the right resp. left edge.

We choose the interval of admissible forces $[\frac{F^*}{2}, F^*]$. The corresponding parabolas cut on the side of the square a line of length

$$\frac{F^* - \frac{F^*}{2}}{2} b(m-b) \geq \frac{F^*}{4} b(d-b) = \frac{kb(l-b)}{36l}$$

Surely, if

$$\frac{1}{2}\frac{kb(l-b)}{36l} \geq N\alpha 2\rho,$$

there is a parabola that does not intersect a negative obstacle. Hence,

$$\rho \leq \frac{kb(l-b)}{144\alpha Nl}.$$

We see that the assumptions on ρ from (a) and (b) are automatically fulfilled.

Conclusion

Theorem

Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ be a non-negative function with

- $\varphi(x) \geq 1$ *if* $x \in [-1,1]^2$,
- $\varphi(x) = 0$ *if* $|x| \geq \alpha$ *for some* $\alpha > \sqrt{2}$.

Suppose that we have two Poisson point processes with parameters $\lambda^\pm$ with $(x_j^\pm, y_j^\pm)$, $j \in \mathbb{N}$ being the corresponding positions of random points. Let us define for every $k \in (0,1]$ and $\rho > 0$:

$$f^{k,\rho}(x,y,\omega) := \sum_j \frac{2k}{\rho}\ \varphi\Big(\frac{x - x_j^+(\omega)}{\rho}, \frac{y - y_j^+(\omega)}{\rho}\Big).$$

Then there exist $\rho = \rho(k) > 0$ and $F = F(k) > 0$ such that a.s. there exists a function $v : \mathbb{R} \times \Omega \to \mathbb{R}$ that satisfies

$$v''(x) - f^{k,\rho(k)}(x, v(x)) + F(k) \leq 0$$

in the viscosity sense and $d((x, v(x)), (x_j^-, y_j^-)) > \alpha\rho(k)$ for all $j \in \mathbb{N}$.