The spin modulus in the context of Riemannian geometry on the general linear group

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A closer investigation of the geodesic distance on the special linear group $\mathrm{SL}(n)$ and the general linear group $\mathrm{GL}(n)$ has recently been motivated by applications in nonlinear continuum mechanics. In particular, the geodesic distance with respect to left-invariant metrics which are right-invariant under actions of the special orthogonal group $\mathrm{SO}(n)$ has been considered in the context of finite-strain plasticity [2] and nonlinear hyperelasticity [3]. Such a metric g on $\mathrm{GL}(n)$ is of the form [1]

$$g_A(X,Y) = \langle A^{-1}X, A^{-1}Y\rangle_{\mu,\mu_c,\kappa}\,,$$

where the fixed inner product

$$\langle X,Y\rangle_{\mu,\mu_c,\kappa} = \mu\,\langle \operatorname{dev}_n \operatorname{sym} X, \operatorname{dev}_n \operatorname{sym} Y\rangle + \mu_c\langle \operatorname{skew} X, \operatorname{skew} Y\rangle + \tfrac{\kappa}{2}\operatorname{tr}(X)\operatorname{tr}(Y)$$

on the tangent space $\mathfrak{gl}(n) = T_{\mathbb{1}}\,\mathrm{GL}(n) = \mathbb{R}^{n\times n}$ at the identity is determined up to three parameters μ, κ, μ_c. In the context of continuum mechanics, the two parameters μ and κ can be identified with the *shear modulus* and the *bulk modulus*, respectively. The additional parameter μ_c on the other hand, dubbed the *spin modulus*, is connected to *plastic spin* in finite-strain plasticity [2] and to the *Cosserat couple modulus* [3] in generalized elasticity.

In order to investigate the specific role of the spin modulus in this setting, we discuss the effect of μ_c on the $\mathrm{GL}(n)$-geodesic distance between pure rotations $Q_1, Q_2 \in \mathrm{SO}(n)$. In particular, we show that this distance is fully determined by the spin modulus for $\mu_c \leq \mu$, in which case the $\mathrm{GL}(n)$-geodesic distance between rotations is identical to their well-known $\mathrm{SO}(n)$-geodesic distance with respect to the classical bi-$\mathrm{SO}(n)$-invariant Riemannian metric on $\mathrm{SO}(n)$ weighted with μ_c. However, we also show that for $\mu_c > \mu$, this equality no longer holds. The obtained results can be used to establish upper bounds on the geodesic distance between arbitrary $F_1, F_2 \in \mathrm{GL}^+(n)$ for the limit case $\mu_c \to \infty$, providing further insight into an earlier conjecture by Mielke [2, p. 86].

References

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